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THE
SCIENCE
OF
ARITHMETIC

THE

SCIENCE OF ARITHMETIC:

A SYSTEMATIC COURSE OF
NUMERICAL REASONING & COMPUTATION,

WITH VERY NUMEROUS EXERCISES.

BY

JAMES CORNWELL, PH.D.

AND

J. G. FITCH, M.A.

NEW AND REVISED EDITION, WITH ADDITIONS

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PREFACE.

THIS book differs from others bearing a similar title in several important particulars.

I. The investigation of the principle on which a rule of Arithmetic depends always precedes the statement of the rule itself.

II. Every process employed in the solution of questions is referred to some general law or truth in the theory of numbers. Thus the operation of "carrying" is explained in § 25, which illustrates the axiom, that—We do not alter the value of any quantity if what we take from one part we add to another.

III. Such general truths are all distinctly stated and printed in italics. If self-evident they are illustrated by simple numerical examples; if otherwise, short demonstrations are added; and in every case the truth itself is enunciated in a concise symbolical form. So, also, the method of equal additions, which is employed in the ordinary mode of working Subtraction, is explained numerically and otherwise,* and shown to be dependent on the axiom, that—We do not alter the difference between two unequal quantities if we add equal sums to both. And this truth is again expressed symbolically in the following way:—

General Formula. If $a - b = x$; then

$$(a + c) - (b + c) = x.$$

---

* §§ 44 and 45.

IV. The theory of Decimals, and rules for the solution of money questions by the decimal method, are placed earlier in the course than usual, and are thus made available throughout the rules of Proportion and Interest.

V. The logical relations of the several parts of Arithmetic are clearly marked by their arrangement. For example, Reduction is not treated as a separate rule, but so much of it as belongs to Multiplication falls under that head, while the rest takes its proper place as one of the practical applications of Division. Interest and Discount, and the kindred rules, are grouped together as illustrations of the doctrine of Proportion; and Practice is treated as a branch of Fractional Arithmetic.

VI. The tables of Foreign Currency, and of English Weights and Measures, are accompanied by an explanation of the French metric system, of the origin of the several standards in common use in England, and of the causes which have led to their diversities and irregularities.

Many important advantages would accrue to beginners, as well as to advanced students, if Arithmetic were regarded more as a branch of mathematical science, and less as a mere system of practical rules. The art of computation is undoubtedly of much value in the business of life; but the habit of investigating the principles on which this art is based is not of inferior importance. The first gives to the student a mastery of figures which will be serviceable in commercial and scientific pursuits: the second tends to concentrate his attention; to induce habits of patient abstraction and accurate thought; to familiarize him with the laws of reasoning, and to compel him to examine well the grounds of every inference he draws. Such habits as these will be invaluable in every pursuit and duty of life, for they will help to make him a sounder and more modest reasoner, and therefore a wiser man.

The value of the exact sciences as instruments of mental discipline has long been recognised. To omit them from any scheme of instruction, however humble, is to allow an important class of the mental faculties to remain untrained. In the limited curriculum of our ordinary schools Arithmetic holds a place analogous to the Mathematics of a university course. It is the only one of the pure sciences usually admitted into such a school, and the only instrument there available for severe and systematic logical training. To degrade Arithmetic into a mere routine of mechanical devices for working sums is, even in a school for young children, to commit as grave and mischievous a mistake as if our university professors were to permit the rules of mensuration to supersede the study of Euclid, or to displace the rigid analysis of the calculus and the higher trigonometry in order to make room for land surveying, the rules of navigation, or the construction of tide tables.

It is only when viewed in this higher aspect that Arithmetic can become an efficient instrument for disciplining the judgment and improving the mental powers. Indeed, it has no right to be called a Science at all, so long as it is limited to "ciphering" on a slate or paper, and does not include a systematic acquaintance with principles as well as rules. To promote such a knowledge of principles something more is necessary than a theoretical treatise on the one hand, or a book of rules, with explanations appended, on the other. Throughout this work, therefore, the principle is in every case first explained and illustrated, and then the rule is shown to follow from it naturally and necessarily. For example, the rules for the extraction of the Square and Cube roots are made to depend entirely upon the theory of Involution. When a pupil has become familiar with the law for the formation of the second or third power,* he will scarcely need to be told the rule for the inverse process of extracting either of those roots; while, on the other hand, without an acquaintance with that law the rule must ever appear to him to be arbitrary and unmeaning.

Besides the explanations necessary to aid the comprehension of the ordinary rules, the plan of this work comprises special exercises on the properties of numbers, demonstrations of some abstract propositions of Arithmetic in a form adapted for repetition by the student,† and questions at the end of each chapter, intended especially to test the pupil's knowledge of the theory of numbers. It will also be observed that one example at least is appended to each rule, in which every part of the process is analyzed, and the separate value of every line of figures clearly shown. These features of the work will, it is hoped, be of use as suggesting some new and not unprofitable methods of exercising the minds of pupils in connexion with this study. For example, a sum when finished may occasionally be taken as the subject of examination and analysis, in the same manner as a sentence in grammar is used for parsing and construing. The exact meaning and value of every figure should be investigated. Every process should be justified by reference to some axiom or proposition, in the same way as a grammarian refers every detail in the construction of a sentence to some definite rule; and if the student be sufficiently advanced, it is desirable, by way of giving neatness and finish to his knowledge of a truth, that he should be able to recognise and interpret it when he meets it in the symbolical form, or "general formula."

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* As enunciated in §§ 414 and 463.

† See Division of Fractions, p. 125, *et passim.*

Although the main object of the authors has been to furnish a systematic and coherent exposition of the theory of Arithmetic, yet the practical uses of the science have been scrupulously kept in view. A comparison of this work with those which are specially valued on account of the abundance and character of the examples, will show that it comprehends everything that is usually regarded as practical or valuable in Arithmetic, in addition to those reasonings and formulæ by which the theory of numbers has been elucidated. The examples have been chosen with especial reference to the pursuits of a commercial people; the subjects of statistics, insurance, stocks, and interest have received particular attention, and an unusual number of business questions occur, both throughout the work and in the Appendix.

Some explanation may seem necessary of the word "axiom," as used at the head of some general propositions and not of others. Without attempting to determine how far even the simplest truths of science are axiomatic and independent of experience, it will suffice to say that the word has been generally applied to such numerical theorems as seemed self-evident from the terms in which they were enunciated, and were not to be deduced as inferences from any previous statement in the work.

Many of the questions are selected from the Cambridge and London University papers, and from those proposed to teachers who have been candidates for certificates of merit. Some specimens are added of examination papers set at the Oxford and Cambridge Local Examinations. The student who masters the reasonings and becomes familiar with the rules and exercises of this book will, as far as Arithmetic is concerned, be competent to pass with credit the ordinary examination for the degree of B.A. at any of the universities.

It only remains to caution students against attempting to make a very hasty progress through this work. The sort of proficiency in Arithmetic which is obtained by evading its difficulties, and hurrying on to the advanced rules, is very worthless. It is fatal to studious habits, and is not even available for practical purposes. A few principles, thoroughly sifted and understood, will be found to form a better substratum for future mathematical or commercial attainments than all the rules of a book if studied apart from those principles. To those departments of this subject which especially require thought and examination, Bacon's aphorism applies with remarkable force:—"Non inutiles scientiæ existimandæ sunt, etiam quarum in se nullus est usus; si ingenia acuant et ordinent."

TABLE OF CONTENTS.

Page
INTRODUCTION 13

LANGUAGE OF ARITHMETIC.

Numeration and Notation 15
Questions for Examination 21

ADDITION AND SUBTRACTION OF INTEGERS.

Sect. I. Simple Addition 22
Methods of Proving Addition 28
II. Compound Addition, or the Addition of Concrete Quantities 29
III. Subtraction 32
Methods of Proving Subtraction 38
IV. Compound Subtraction, or the Subtraction of Concrete Numbers 38
V. ** Supplementary Section on Addition and Subtraction. Theory of Sums and Differences 42
Questions on Addition and Subtraction 43

MULTIPLICATION AND DIVISION.

Sect. I. Simple Multiplication 44
II. Compound Multiplication, or the Multiplication of Concrete Numbers 54
III. Abbreviated Methods of Multiplication 57
IV. Descending Reduction 60
V. Simple Division 62
VI. Long Division 68

Page
VII. Compound Division, or the Division of Concrete Quantities 71
Methods of Proving Multiplication and Division 76
VIII. Ascending Reduction 78
IX. ** Supplementary Section—Multiplication and Division. Theory of Factors, Products, and Quotients 80
Questions on Multiplication and Division 87
Miscellaneous Exercises on the Arithmetic of Integers 89

PRIME AND COMPOSITE NUMBERS.

Sect. I. Measures and Multiples 92
Greatest Common Measure 95
Least Common Multiple 101
II. ** Special Properties of Numbers 104
Questions on Prime and Composite Numbers 107

FRACTIONAL ARITHMETIC.

Sect. I. Notation of Vulgar Fractions 108
II. Addition and Subtraction of Vulgar Fractions 119
III. Multiplication of Vulgar Fractions 121
IV. Division of Vulgar Fractions 124
V. Reduction of Vulgar Fractions to others of different denominations 127
VI. Continued Fractions 129
VII. Miscellaneous Applications of Vulgar Fractions 133
Practice 137
Questions on Vulgar Fractions 141
Miscellaneous Exercises on Vulgar Fractions 142
VIII. Decimal Fractions 144
Recurring Decimals 152
IX. Addition, Subtraction, Multiplication, and Division of Decimal Fractions 158
Addition of Decimals 158
Subtraction of Decimals 160
Multiplication of Decimals 162
Division of Decimals 164
** Contracted Methods of Multiplication and Division 169
X. Application of Decimals to Concrete Quantities 172

Page
XI. Decimal Money 176
Questions on Decimal Fractions 181
Miscellaneous Exercises on Fractional Arithmetic . 182

RATIO AND PROPORTION.

Sect. I. Theory of Proportion of Abstract Numbers . . . 186
II. ** Supplementary Section on the Theory of Proportion 194
III. Rule of Three, or Simple Proportion 196
IV. Compound Proportion, or the Double Rule of Three 207
V. Interest, Discount, &c. 213
Simple Interest 213
Compound Interest 218
Discount 220
Stocks and Shares 225
Miscellaneous Applications of the term Per Cent. . 230
Questions on Ratio and Proportion 234
Miscellaneous Exercises on Proportion 235

INVOLUTION AND EVOLUTION.

Sect. I. Involution to the Second Power, or Formation of the Squares of Numbers 243
II. ** Theory of Numerical Squares and Products . . 247
III. Evolution from the Second Power, or Extraction of the Square Root of Numbers 252
Extraction of the Square Root by Approximation . 255
IV. Involution to the Third Power, or Formation of the Cubes of Numbers 262
V. Evolution from the Third Power, or Extraction of the Cube Roots of Numbers 265
Questions for Examination 272

APPLICATION OF ARITHMETIC TO GEOMETRICAL MEASUREMENTS.

Sect. I. Duodecimals 274
Mensuration of Parallelograms 274
II. „ Triangles 278
III & IV. „ Circles 279
Parallelopipeds 282
V. „ Spheres 284
„ Cylinders 284
„ Cones 284

Page

PROGRESSION AND LOGARITHMS.

Sect. I. Progression by Equal Differences, or Arithmetical Progression 286
II. Progression by Equal Ratios, or Geometrical Progression 293
III. Logarithms 300
Application of Logarithms—I. Multiplication, &c. . 308
" " II. Interest, &c. . . . 311
" " III. Annuities 313
Miscellaneous Exercises on Logarithms 318
Questions on Progression and Logarithms 320

APPENDIX.

A.—MONEY—WEIGHTS AND MEASURES.

Sect. I. The Unit or Standard of Measurement 321
II. Measures of Value—Money. 322
Tables of Foreign Currency. 324
III. Measures of Length 326
French Linear Measure 328
IV. Measures of Area or Surface 329
V. " Solidity and Capacity 330
VI. " Weight 332
VII. " Time 334
B.—VARIOUS METHODS OF NOTATION 336
MISCELLANEOUS QUESTIONS 341
SPECIMENS OF EXAMINATION PAPERS SET AT THE OXFORD AND CAMBRIDGE LOCAL EXAMINATIONS 348
ANSWERS TO THE EXERCISES 357

The Sections which are distinguished by a double asterisk (**) occupy their proper positions in the book, as far as their logical relation to the rest determines it; they are nevertheless not absolutely necessary for the comprehension of the subsequent rules, and may be omitted by a student who is reading the work for the first time.

THE SCIENCE OF ARITHMETIC.

1. The Science of Arithmetic treats of NUMBER.*

The first and simplest notion we must obtain in the Science of Arithmetic is that of UNITY, or oneness.

2. Any object, or any magnitude considered singly and without reference to its parts, is called a Unit.

Thus when we perceive and speak of one man, a horse, or a stone, we have before us the idea of unity, and are said to be thinking of a unit.

3. Any collection of units of the same kind is called a Number.

When we speak of a crowd of persons, a heap of stones, or ten books, or five hours, we have before us the notion of number. These expressions, however, would not be understood if the notion of *one* person, *one* stone, *one* book, or *one* hour was not in the mind first. Unity, therefore, is the one magnitude with which all magnitudes of the same kind are compared.

If we look at a row of trees or a group of children, we cannot help considering one tree and one child as the unit or standard of reference, when we attempt to number them ; but in expressing the length of a line, or the size of a field, or the duration of time, we are left to choose any unit we please. Thus the same length of line may either be expressed as two yards or as six feet ; the number used being dependent on the sort of length which has been selected as the unit. In any magnitude, such as time, space, length, or weight, which is capable of *continuous* increase, the choice of the unit is arbitrary. But when we use number to describe collections of *separate* objects of the same kind, one of those objects is always taken as the unit.

---

* It forms a branch of the Mathematics, which include also Algebra, Geometry, Trigonometry, and several other sciences. The first of these treats of the same subject as Arithmetic, but in a wider sense. The remaining two investigate the properties of space. All branches of Mathematics relate to magnitude and the modes of measuring it.

4. A CONCRETE NUMBER is a number of objects of any kind, as *five apples, ten pounds, sixteen men.*

An ABSTRACT NUMBER is a number unconnected with objects, as *five, ten, sixteen.*

The notions of number which the experience of our senses gives us, are always *concrete*, and are made up of two elements, the notion of number and the notion of the objects. The mind, however, can separate the conceptions of number from those of objects, and when this is done we have before us the *abstract* notion of number.

The least instructed person who looks at the heavens and sees six bright stars, and also into a garden and sees six roses, cannot help understanding that although there is no resemblance between a star and a rose, yet there is some resemblance between the two notions of the *six* stars and the *six* roses. These two compound ideas have something in common, and that something is the abstract idea of number, which is expressed in our language by the word six.

5. The Science of Arithmetic is intended to give clearness and accuracy to our notions about number. We may think about number without the aid of Arithmetic, but always in a vague and indistinct way ; as when we see two heaps or collections of objects, and observe that the one contains *many* and the other *few*, or that the one is a *larger* heap than the other ; but if we wish to ascertain *how* many or *how* few either contains, or *by how much* the size of the one exceeds that of the other, we are compelled to employ the symbols and the processes of ARITHMETIC.

LANGUAGE OF ARITHMETIC.

6. The language employed in Arithmetic may be considered in two ways :—I. As consisting of a set of *words* or *sounds* intended to express number ; and II. As composed of *written* characters or *signs.*

Some rude nations have the former without the latter; they have language about number which can be addressed to the ear, but have no figures or symbols which can be presented to the eye.

It must be remembered that the notion of any number is necessarily the same with all persons, but the mode of expressing it, either in spoken or written words, or in characters, may be and is different with different nations. Thus in looking at a herd of oxen, all persons would have precisely the same notion of their exact number, but while we should call the number *twelve,* a Frenchman would call it *douze,* and a German *zwölf,* and in the employment of characters modern nations would represent by 12 what the Romans represented by XII. The *notion* of number itself is *essential,* while the mode of *expressing* it is *arbitrary,* whether in writing or speech.

7. NUMERATION is the art of numbering, or of expressing number *in words.*

Observation.—The various collections of units which we meet with and wish to express are so numerous, that if we had a name for every such collection, Arithmetic would require more words than all the sciences put together. There is in fact no limit to our power of making new numbers, for however great may be the collection of units which a person may think of, it will still be possible for him to think of a greater. All nations have a few separate words or names for some particular collections of units, and express other numbers by putting together these words in different ways.

8. We have in common use fifteen distinct words only, viz. :—

one		*six,*	five and one	*eleven,*	ten and one
two,	one and one	*seven,*	six and one	*twelve,*	eleven and one
three,	two and one	*eight,*	seven and one	*hundred,*	ten tens
four,	three and one	*nine,*	eight and one	*thousand,*	ten hundreds
five,	four and one	*ten,*	nine and one	*million,*	one thousand thousands

All other numbers or collections of numbers are expressed by combining some of these words in various ways.

9. AXIOM I.—*All the parts of a number put together make up the whole, in whatever order they may be taken.*

Demonstrative Example.—The first word in our language which illustrates this principle is *thirteen;* taken to pieces it means *three* and *ten.* We might have called it ten-three, or nine-four, or seven-six, or six-five-two, or three-four-six, or by any other name which mentioned all its parts. But it is more convenient to divide all numbers in exactly the same way.

10. The number TEN has been chosen as the basis of all our calculations, and we consider all numbers as made up of tens or some collections of tens. Ours is therefore called the DECIMAL SYSTEM.*

Observation.—It is probable that this practice, which prevails in nearly all parts of the world, arose from the use of the ten fingers in simple calculations. In counting on the fingers (as children often do now) we can go on easily until we come to ten, but are obliged to use some other contrivance for all higher numbers. If we had had twelve fingers instead of ten we should probably have acquired the habit of counting by twelves, and of considering all large numbers as composed of so many twelves.

It would have been just as easy, had we been accustomed to it, to consider numbers as made up of fours, or nines, or in any other way than of tens: habit alone makes our present method seem simple. If, for example, we hear of a number consisting of nine, seven, and eight, we have not at first a clear notion of what it means, but we add the parts together and call the whole twenty-four. Yet it is not now before the mind as a whole, but in three parts (two tens and four), and it was in three parts before (nine, seven, and eight). Of all the possible methods of considering the number, that which breaks it up into tens is most convenient merely because we are accustomed to it.

EXERCISE I.

☞ Write out in other words what is meant by the following expressions:—

Example.—Forty-five = four tens and five = six sevens and three.

Six-teen; Twenty-seven; Eighty; Forty-five; Three hundred and sixty; Five thousand four hundred and seventy-four; Five hundred and ninety-five; Fifty-eight; Twelve; Seventy-nine; One million six hundred thousand; Twelve hundred and ninety; Three thousand and forty-five; Eight hundred and sixty; Three millions; Eight thousand four hundred.

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* From *decem,* ten (Latin).

11. NOTATION is the art of expressing numbers in *written* characters.

Our Notation contains *nine* written characters or significant figures.

For the reason suggested in (10) these characters are often called *digits*.

The written language, like that which we speak, uses a few symbols to express small numbers, and collects some of these together in a certain order to express larger ones.

The only characters or figures in use are the following:[*]—

1, One
2, Two
3, Three
4, Four
5, Five
6, Six
7, Seven
8, Eight
9, Nine[†]

When any one of these figures is found alone, and in this form, it only stands for one of the first nine collections of units, and is said to indicate *unity of the first degree.* All numbers above nine are, however to be represented by the same figures ; so sometimes we wish the figure 1 to mean one ten ; 2 to mean two tens ; 3, three tens, and so on. Whenever the nine digits are used in this sense they are said to indicate *unity of the second degree.* At other times we require to make use of the figures 1, 2, 3, &c., to mean 1 hundred, 2 hundred, &c. ; and in this case they refer to *unity of the third degree.* When they are employed to signify thousands, they are *units of the fourth degree ;* when a figure is used to mean tens of thousands, it represents *unity of the fifth degree ;* hundreds of thousands, *unity of the sixth degree ;* millions, *unity of the seventh degree*, and so on. We therefore want some contrivance to show when the figures are used in one sense and when in another.

---

[*] That is, irrespective of the nought or cipher, which does not represent any number, but only gives a determinate significance to the nine characters, as explained in (16).

[†] It will be observed that our written language is not quite so copious as the spoken, and does not exactly correspond to it. Thus the word *eleven* is a simple name which to the ear suggests no thought of combination, but the figures 11 represent composition, and inform us that the number consists of two parts, viz., one collection of units and one. In French, the difference between the written and the spoken language is much wider. The number *ninety-two* is written in that language as in English (92), and represents 9 collections of ten each and 2 units. But it is expressed by the words (quatre-vingt-douze), which signify four twenties and twelve. Other words used to express number in French are founded on a division into twenties, while the written characters treat all collections of units as consisting of tens.

12. Suppose we use the simple digit when we wished it to represent unity of the first degree, and placed a mark over it thus (′) when we wished it to stand for tens, then the expression $17'$ or $7'1$ would mean 1 and 7 tens, or 7 tens and 1. Either form would serve the purpose. So $6'5$ would mean 6 tens and 5, and would represent the same number as $56'$. In like manner, unity of the third degree might be represented by a digit with two strokes above it; thus $5''$ would mean 5 tens of tens, or 5 hundreds: thousands, or tens of hundreds, might be expressed by digits with three strokes above them ($6'''$), and all higher numbers in like manner. If such an arrangement existed, the expression $4'''6''82'$ would mean, 4 units of the fourth degree (4 thousands), 6 units of the third degree (6 hundreds), 8 units of the first degree (8), and 2 of the second (2 tens). As the value of each figure would then be known by the number of accents above it, the digits might be put down in any order without altering their meaning. Thus $2'4'''86''$, or $84'''2'6''$, would represent the same collections of numbers.

13. It would be possible also to represent larger numbers by the help of nine simple digits by using the letters A, B, C, &c., to show the different meanings of the digits. Thus 7A might mean 7 tens, 7B seven hundreds, 7C seven thousands, and so on. On this plan the expression 8B, 6D, 5, 4A, would mean 8 units of the third degree (8 hundreds), 6 of the fifth (6 tens of thousands), 5 of the first (5), and 4 of the second (4 tens). The number which we call 5 thousand 4 hundred and 9 tens and 4, would be written 5C, 4B, 9A, 4, or 4B, 4, 9A, 5C; for as each figure's value would be shown by the letter which followed it, they might be placed in any order without altering the meaning of the whole expression.

Again, such expressions as 2^3, 4^2, 7^5, could easily have been employed to signify 2 units of the third degree, 4 units of the second degree, and 7 units of the fifth degree. We use these forms now with another meaning (see Involution 407), but they would have served equally well for this purpose.

Many other plans might be devised for representing any numbers whatever on the decimal system, by the help of nine digits or significant figures; all that is essential to a complete system of Notation, adapted to an infinite variety of numerical values, being—

I. A limited number of significant characters.

II. An arrangement showing when these characters have different meanings, and what these meanings are.

14. The value of each separate figure used in our Arithmetic is shown by its POSITION only.

For example, that which in our spoken language is called Ten, is written as the figure 1, with one other figure to its right. In like manner that which is expressed by the word Hundred, is represented in writing by the figure 1, with two others to its right. By this plan it is unnecessary to attach any sign to a 3 or a 5 to show that it should mean 3 tens or 5 tens of tens, the meaning of each figure being shown simply by its place.

15. The units of the several degrees (11) have their value represented by the place in which they stand, beginning at the right hand, as follows:—

A number meaning Units only, is a unit of the first place.
.... Tens .... second ...
.... Hundreds .... third ...
.... Thousands .... fourth ...
.... Ten-thousands .... fifth ...
.... Hundred-thousands ...... sixth ...
.... Millions .... seventh...

Thus in the line of digits—6666666,
which we read, "Six millions, six hundred and sixty-six thousands, six hundred and sixty-six,"—

every 6 has a different value; the first on the right hand means simply 6; the second has one figure on its right, and means 6 tens; the third has two figures after it, and means 6 hundreds; the fourth, having three to its right, means 6 thousands; the fifth, 6 tens of thousands; the sixth, 6 hundreds of thousands; and the last, which has six figures to its right, means 6 millions. Each of the figures will be seen to mean ten times more than that on its right.

EXERCISE II.

☞ In the following collections underline all the figures which mean—

(a). Tens. 575; 64; 8297; 48623; 59; 161; 3287; 15.
(b). Hundreds. 287; 35698; 4523; 889621; 3391; 178; 39625.
(c). Thousands. 2896; 5832145; 627489; 82713; 9862; 59418.
(d). Tens of thousands. 17698; 253710; 9168542; 217906.
(e). Millions. 827463857; 94728547; 832749683; 7283471.

16. The Cipher * (0) has no meaning in itself, but is only useful in determining the place of other figures.

To represent the number Four Hundred and Five in writing two figures only will be needed, one to signify four hundred, and the other five; but if these two figures are set down together thus, 45, the 4 will be mistaken for 4 tens, being only in the second place. To mean four hundreds, the figure 4 should have two figures to its right (15); the cipher is therefore put in the place usually given to tens to show that the number is composed of hundreds and units only, and that there are no tens. Four hundred and five is therefore written 405.

EXERCISE III.

☞ Write out in words the separate value of every figure in the following expressions:—

Example.—56812; fifty thousand, six thousand, eight hundred, one ten, and two, or fifty-six thousand eight hundred and twelve.

2305; 806; 7095; 20300; 457298; 627421.
33911; 427816; 9032804; 8271096.
32745841; 72918; 13; 5172; 840; 621.

17. *In reading* figures it is usual to break up the whole into periods of three figures each, beginning with the unit.

Observation I.—We have a name (tens) for all the units of the second place, another (hundreds) for units of the third place, and one (thousands) for units of the fourth place. But for units of the fifth place we have no new name, but are obliged to combine two of the words already employed—they are called *tens of thousands.* So, also, we have no special name for units of the sixth place, for we call them hundreds of thousands. But for units of the seventh place (thousands of thousands) we use the word millions. The word hundreds is used twice when we read a line of six figures, and three times when we read a line consisting of nine figures. Thus the figures 763,842,517 are read, seven *hundred* and sixty-three millions, eight *hundred* and forty-two thousands, five *hundred* and seventeen. That is, we speak first of hundreds of millions, secondly of hundreds of thousands, and lastly of simple hundreds (hundreds of units). It is, therefore, sometimes the practice to mark off the last three figures in the row and consider them as units, the next three as thousands, and the remainder as millions. But this is quite arbitrary, and is merely an arrangement of convenience.

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* *Cipher*, from an Arabic word, meaning empty or void.

Observation II.—A thousand millions are sometimes called a billion, or in French, a milliard, but these words and others for higher combinations, as trillion and quadrillion are practically unused.

EXERCISE IV.

☞ Point off the following lines of numbers, and read them in periods of three :—

17094632; 508704602; 290782; 5069413.
8274169325; 2748629174; 3072; 89621543.
728034; 6195; 83274; 409608.
30729; 8504640; 3270562; 92807.
50963; 827041; 2031.
83265709125; 219873654213; 319680259417.

EXERCISE V.

☞ Represent in figures the following expressions :—

(1.) Nine hundred and eighty; forty thousand and two.

(2.) Seven thousand six hundred; eighty one thousand four hundred and two.

(3.) Two hundred and fifty; five millions four thousand and seven.

(4.) Eight thousand six hundred; twenty-four thousand and five.

(5.) Twelve hundred and fifteen; six thousand four hundred and ten.

(6.) Nine hundred and eighty-one; eighteen millions and six.

(7.) Four hundred and thirteen; five hundred.

Questions for Examination.

Define Arithmetic, Number, a Unit. What is meant by the arbitrary selection of a unit, and when is it employed? Distinguish between abstract and concrete numbers. What is to be understood by a system of Numeration? What by a system of Notation? What are the necessary requisites of every system of Notation? What truth in Arithmetic forms the foundation of all such systems? Give examples.

What is the meaning of the word Decimal? How is it applied in Numeration? When is a system said to be Decimal? To what circumstance is the choice of the decimal method commonly attributed? Describe any possible method of representing numbers decimally which differs from our own.

What is the distinguishing feature of our plan of decimal notation? How are we to make any given number represent hundreds, thousands, millions? Take a number, 8 for example, and say how it must be placed so that it shall mean 8 hundred thousand, or 8 tens of millions.

Describe the use of the cipher in Notation. Give examples of lines of figures containing two, three, four ciphers, and read them. What practical advantage arises from pointing off figures in groups of three?

ADDITION AND SUBTRACTION.

SECTION I.—SIMPLE ADDITION.

18. ADDITION is the process of finding one number which shall be exactly equal to two or more other numbers; *or* of placing several collections of units together and finding one expression which represents the amount of them all.*

For instance, when we propose this question, "How many will 5 apples and 12 apples and 6 apples make if all put together?" or, "If 25 sheep are in one field and 14 in another, how many would there be if they were all turned into one?" we ask a question in Addition, for it is only by *adding* or combining the several numbers mentioned that we can obtain the answer we seek.

19. The answer to any question in Addition is called the SUM of the numbers mentioned in the question.

Thus, because four and five make nine, nine is the *sum* of four and five.

20. *Signs.* +, called *plus*, is the sign of addition.

Thus $7+6$ (read seven *plus* six) means seven *and* six.

=, called *equals*, is the sign of equality.

Thus $6+5=11$ (six *plus* five *equals* eleven) means that the six and the five taken together are the same as the number eleven.

Or, $a+b+c=d$ (read a plus b plus c equals d) means that the *sum* of the quantities called a and b and c makes up the quantity called d.

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* *Preliminary Mental Exercise.*—It is useful for the student to be practised well in the addition of small numbers before doing sums in writing; *e.g.*, "To 4 add 7 and 7 and 7, &c., or to 5 add 6 and 6 and 6," &c., and so on with each of the nine digits in turn. When all combinations of numbers have been exhausted in this way the pupil will be able to carry any such series up to a hundred very rapidly.

21. AXIOM II.—*If we add all the parts of any numbers together, we add the numbers themselves.*

Demonstrative Example.—The number *fourteen* may either be added to another as a whole; or *ten* may be added, and then *four;* or it may be resolved into $7+5+2$ and added in that form.

So $7+6+5=5+7+6=5+4+3+6$.

General Formula.—If $a+b=x$ and $c+d+e=y$,

then $a+b+c+d+e=x+y$.

22. *Observation.*—We pursue this plan in all the rules of Arithmetic, treating numbers piecemeal, and counting them, part by part, for this reason: *We only know the relations between a few small numbers; these we learn by habit and practice, and it is by considering all larger numbers broken up into such parts as we know, and operating upon these parts successively, that we operate upon the whole.* All Arithmetic consists of contrivances for resolving long or complex operations into a series of simple ones. For we have no power which will enable us to say at once how much two hundred and eighty-seven and seven hundred and fifty-six make when added together. But we can first add the two hundred and the seven hundred, then the eight tens and the five tens, and lastly the seven and the six. Each of these operations is familiar to us by itself; but very few people could leap to the answer at one bound.

EXERCISE VI.

☞ Express each of the following numbers in four different ways decomposing them into parts which are equal to themselves :—

Example.—$25=2$ tens $+ 5 = 18+7 = 10+15 = 16+9 = 12+6+7$.

74; 120; 36; 273; 85; 884; 3795; 24; 83; 127; 29; 65; 128; 230; 472; 191; 654; 37; 86; 94; 72; 230; 719; 685; 32.

In computing, it is desirable at each step not to mention the figures themselves, but only the result of each addition, for example,

$$\begin{array}{r}7\\9\\8\\3\\2\\5\\1\\6\\8\\\hline 49\end{array}$$

It is evident that the sum of these numbers will be the same in whatever order they are placed; hence it is a good plan, after getting an answer to one of these sums, to begin again at the opposite end of the column and add them backwards. Thus, beginning at the foot of the first sum, the learner should not say 8 and 6 are 14 and 1 are 15, &c., but 8, 14, 15, 20, 22, 25, 33, 42, 49. To prove that he is right he may then begin at the top and say 7, 16, 24, 27, 29, 34, 35, 41, 49. The more rapidly he can repeat these numbers in succession, the more likely he will be to obtain the right answer. Slow computers are generally inaccurate.

For examples of other forms of oral exercise, see School Arithmetic, p. 9.

23. AXIOM III.—*Concrete numbers can be added together only when they refer to the same class of objects.*

Demonstrative Example.—5 horses and 7 sheep do not make either 12 horses or 12 sheep. The number 12 cannot properly describe these collections at all, unless we use some word, such as quadrupeds or animals, which applies equally to both horses and sheep, and say, 5 horses + 7 sheep = 12 quadrupeds.

In the same way, whenever any special meaning is attached to any figures, those only can be added together which have the same meaning. In 500, for example, the 5 represents 5 units of the third degree, or 5 hundreds, and in 60 the 6 represents 6 units of the second degree. Now these two numbers cannot properly be put together and said to make 11 of any kind of unit.

24. Hence it is necessary to arrange our sums so that the eye shall immediately perceive the numbers which ought to be added together. Thus if we have £10 9s. 6d. and £7 4s. 2d. and £3 8s. 7d. to add together, it will be convenient to place the pounds in a column by themselves, and the shillings and pence also in vertical lines. In like manner, suppose we wish to tell the amount of the numbers, 794 + 29 + 3057 + 652, it will be convenient to place them in columns also, thus :—

£.	*s.*	*d.*
10	9	6
7	4	2
3	8	7
—	—	—
20	21	15

Thous.	Hunds.	Tens.	Units.
	7	9	4
		2	9
3	0	5	7
	6	5	2
3	13	21	22

Those numbers which represent units of the same degree or denomination are now in the same vertical lines, and by adding up each column by itself it is found that all the parts of the given numbers, taken together produce £20, 21 *shillings, and* 15 *pence,* and 3 *thousands,* 13 *hundreds,* 21 *tens, and* 22 *units.*

Observation.—Since (15) the value of each separate figure depends on the number of digits to its right, it is not necessary to use lines to separate the various columns of numbers. They will easily be distinguished by placing the units of each number in the right hand column, the tens in the second, and *all figures of the same value in the same column.*

25. AXIOM IV.—*We do not alter the value of any quantity, if what we take from one part we add to another.*

Demonstrative Example.—The answer in (24), viz., 3 thousands, 13 hundreds, 21 tens, and 22 units, is not given in the most convenient shape. The 13 hundreds consist of 1 thousand and 3 hundreds, so that 3 thousands + 13 hundreds = 4 thousands + 3 hundreds. Similarly, the 21 tens consist of 2 hundreds and 1 ten; 3 thousands, 13 hundreds, and 21 tens are the same, therefore, as 4 thousands, 5 hundreds, and 10. But the 22 units = 2 tens and 2 units, therefore the whole number—

$$\left.\begin{array}{c|c|c|c} 3 & 13 & 21 & 22 \\ 3000 & 1000 + 300 & 200 + 10 & 20 + 2 \end{array}\right\} = 4 \mid 5 \mid 3 \mid 2$$

That is to say, we have *carried* from the units place 20 and placed it among the tens, as 2; we have carried 20 tens from the second column and placed it among the hundreds, as 2; we have carried 10 from the hundreds and placed it among the thousands, as 1. By thus transferring these parts from one place in the line to another, no alteration is made in the value of the line itself, and the answer is expressed more concisely This process is called *Carrying*.

26. Since (21) the summing up of all the parts successively is the summing up of the whole, in whatever order the several columns or lines of figures are added up, the result must be the same. For example—

Add together 4083, 769, 8402, 1237, 9626, and 8642.

We first arrange the units in a vertical column, then the tens, afterwards the hundreds and the thousands, thus:—

$$\begin{array}{cccc} 4 & 0 & 8 & 3 \\ & 7 & 6 & 9 \\ 8 & 4 & 0 & 2 \\ 1 & 2 & 3 & 7 \\ 9 & 6 & 2 & 6 \\ 8 & 6 & 4 & 2 \\ \hline 30 & 25 & 23 & 29 \\ \hline 32 & 7 & 5 & 9 \end{array}$$

We may first add up the thousands, then the hundreds, then the tens, and lastly the units. This answer, however, wants to be corrected by bringing the tens of each denomination into the column on the left. But if we begin with the units column this second operation may be avoided, for, on finding the amount to be 29, it is the 9 only that remains as units, and the 20 is at once transferred to the next column, under the name of 2 tens. Hence it is *more convenient* always to begin at the right hand.

RULE FOR SIMPLE ADDITION.

27. Arrange the figures so that the units of the same value stand exactly in the same column (24). Add up the figures in the first column on the right[*] (26). If the total is no more than 9, set it down at once under the units. If it be an exact number of tens, place a cipher under the units column, and remove the tens to the next column; but if more, set down the units only, and add the tens to the next column. In the same way add up the tens column, carry the hundreds, if any, to the third column, and place the remaining tens only in the second column. Proceed in this way, till each column has been added up.

EXERCISE VII.

☞ Add together the following collections of numbers :—

1. 123 + 58 + 4094 + 835 + 6294 + 8327 + 5186.
2. 567 + 90 + 48 + 39 + 4728 + 1000 + 6489 + 327 + 4578.
3. 528 + 347 + 269 + 947 + 2586 + 9324 + 82746 + 5372.
4. 23 + 19 + 48 + 61 + 314 + 1000 + 8031.
5. 7 + 19 + 324 + 8 + 160 + 2430 + 29.
6. 12 + 150 + 3987 + 141 + 5 + 67 + 3005 + 498.
7. 13 + 798 + 230 + 647 + 2350 + 826 + 97.
8. 3290 + 574 + 386 + 2074 + 3826 + 5049 + 2786.
9. 325 + 472 + 569 + 8072 + 40961 + 300040 + 713.
10. 58 + 64 + 9721 + 3720 + 5829 + 6874 + 306 + 594.
11. 37045 + 6879 + 3724 + 4562 + 82971 + 37256 + 409.
12. 2304 + 50695 + 28 + 14 + 3972 + 51 + 694 + 2804.
13. 709 + 8304725 + 627 + 81 + 471 + 391 + 2740 + 83.
14. 200 + 371209 + 621 + 30 + 8594327 + 3269 + 942.
15. Add together three millions and forty, two hundred and five, sixteen thousand eight hundred, twelve hundred and fifty-nine, eighty-six, and ten thousand and four.

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[*] It is important that the pupil should not mistake rules of mere convenience for rules which depend on principles. It will, therefore, be well that a few sums should be done in the way described in (26), before the usual method is adopted.

16. Add together four hundred and seventy-three, five hundred and ten thousand, sixteen millions, five hundred and eighty, thirty-seven thousand three hundred and seven.

17. Add together four, five, nine, fifty, three hundred and eight, ten thousand, two thousand and nine, forty-eight, sixteen, and twelve.

18. Find the sum of seventy-eight, two hundred and six, four hundred and eighty-three, twelve thousand three hundred, fifty-four millions, two thousand six hundred and seven.

19. What is the total amount of three hundred and five, seven thousand and ninety, six hundred and forty-seven, eight thousand three hundred and fifty-four, two hundred and eighty-six, four thousand three hundred and eight, seven hundred and forty-five?

20. A man had in his possession several hoards of money. In one place he had 27 gold pieces and 156 silver ones, in another 758 copper coins, 123 of gold, and 287 of silver; in a third, 96 of each; in a fourth, 37 of gold, 29 of silver, and 100 of copper. How many coins of each kind had he, and what was the total number of pieces?

21. The less of two numbers is 527 and the difference 279, what is their sum? If the less be 782 and the difference 156, what is the sum?

22. A corn merchant had four granaries, the first contained 297 quarters of wheat, 563 of barley, and 641 of oats; the second, 8507 quarters of barley, 709 of wheat, and 56 of oats; the third contained 2634 of wheat, 1617 of oats, and 500 of barley; the fourth contained 728 quarters of each. How much of each kind did he possess, and what was the total quantity of grain?

23. How long is it since the year 1491 B.C.? How long since 347 B.C.? since 4004 B.C.? since 2348 B.C.? since 445 B.C.?

24. How many days are there from March 5th to Dec. 19th, inclusive?

25. If A possesses £537, B £29 more than A, C as much as A and B together, and D £185 more than the sum of the other three, what is their total possession?

26. A postman delivered 628 letters on Monday, 510 on Tuesday, 496 on Wednesday, 837 on Thursday, and as many in the last two days as in the three previous days, how many did he deliver in the week?

27. What is the amount of the population of the six counties containing the greatest number of inhabitants, seeing that the following were the returns made at the census of 1871?—Lancashire, 2,818,904; Middlesex, 2,538,882; Yorkshire, 2,436,113; Surrey, 1,090,270; Kent, 847,507; Staffordshire, 857,333.

METHODS OF PROVING ADDITION.

28. To prove a sum is to verify the answer, and to show that no other could be correct.

I. Cast up each line of figures twice, beginning at the foot of the column in one case and at the top of it in the other. It is so unlikely that the *same* mistake should be found in both answers, that when they agree we generally assume them to be right.

29. II. Select any line from the sum, and after the answer has been found, work the whole a second time, omitting that line. If when this line is again added the same answer is found, we conclude that it is the true one.

In the examples *a* and *b* it may be seen that the answer is first found in the ordinary way. Then one line is omitted, which is afterwards added by itself.

a	*b*	*c*
70962	70962	70962
2385	2385	8342
247	247	749
(50689)	3247	1256
3247	5469	39724
5469		82461
	82310	
132999	50689	203494
	132999	23220

30. III. In the example *c* begin at the left hand and cast up the first column instead of the last; thus, 7 and 3 and 8 are 18. But 20 stands underneath ; set down the difference between 18 and 20, for this 2 must have been brought from the column to the right. Again, 2 and 9 and 1 and 8 are 20, but the 2 tens of thousands brought from the left, and the 3 thousands, make 23 ; set down 3 underneath, for this is the difference between the 20 thousands in the column and the 23 thousands in the answer. Again, the sum of the hundreds column is 32 ; take this from 34 and set down the remainder 2 under the 4. Now cast up the tens column alone ; they amount to 27. But because the 2 hundreds already set down, and the 9 tens in the answer, make 29 tens, set down the difference between 27 and 29 under the tens. As the sum of the units column is exactly 24, or 4 + 2 tens, the answer is proved to be right.

31. IV. Add up the digits in each line, cast out the nines, and place the remainders to the right. Do the same with the answer, and if this last remainder is the same as is obtained by casting out the nines from the sum of the other remainders, the answer is probably right.*

Example.—	796324	.	4
	28761	.	6
	103587	.	6
	928672	.	7

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* The reason of this rule is shown in Section 175.

SECTION II.—COMPOUND ADDITION.

32. Suppose in the following sum the figures in the first column

$$\begin{array}{ccc} 7 & 5 & 3 \\ 8 & 2 & 7 \\ 4 & 6 & 2 \\ 3 & 9 & 6 \end{array}$$

on the left, 7, 8, 4, 3, meant pounds instead of hundreds of units, those in the next column shillings, and those in the third column pence; the same rules that have been already stated will apply exactly. But here the value of the figures in the three columns, though dependent as before on local position, is not regulated by the decimal system, but by the number of pence in a shilling, and of shillings in a pound. In the top line, for example, the 5 means twelve times more than if it had been one place to the right, and the 7 means twenty times more than if it had stood in the second column. When, therefore, we add together the numbers in the third line (6, 2, 7, and 3), and find them to be 18, we must consider this number of pence, not as 10 and 8, but as 12 and 6, the 12 pence being one shilling. This 1 may, therefore, be carried to the place of shillings, and the 6 only set down to pence; similarly, the column of shillings amounts to 23, but because 23 shillings make £1 and 3 shillings, the 3 only must be placed under the shillings, and the 1 transferred to pounds.

33. As, ordinarily, columns of figures are arranged decimally, that is, each figure in a column is worth ten times as much as is that to the right, it is necessary, when this arrangement is departed from, to mark it by some particular signs. This is done by placing the name of the quantity, or its contraction, above each column, and by separating columns of different values by marks (,,).

So the sum would not be set down as above, but thus :—

$$\begin{array}{ccccc} \pounds & & s. & & d. \\ 7 & ,, & 5 & ,, & 3 \\ 8 & ,, & 2 & ,, & 7 \\ 4 & ,, & 6 & ,, & 2 \\ 3 & ,, & 9 & ,, & 6 \\ \hline 23 & ,, & 3 & ,, & 6 \end{array}$$

34. Whenever sums of money, or quantities of weight, or length, or other magnitudes expressed by concrete numbers, require to be added together, it is only necessary to know the meaning of the several terms employed (pound, yard, ton, mile, &c.), and how much of any one is contained in one of another name. This knowledge may be obtained from the tables in the Appendix, which should be learnt by heart.

RULE FOR COMPOUND ADDITION

35. Place those numbers which refer to the same quantities in separate columns (24); add the numbers in each column by themselves, beginning with those of the lowest value (26), and transfer into the next column as many of the less as make one or more of the greater, placing only the remainder under the first; proceed in the same way until the numbers of the highest denomination are reached, when the process is as in simple numbers.

Example.—Add together—

tons.	cwt.	qrs.	lbs.	oz.
7	3	2	27	0
5	11	3	19	5
6	2	2	24	9
5	8	3	9	15
2	13	0	5	12
25	37	10	84	41
27	0	1	2	9

Exercise VIII.

☞ Work out the following Addition sums by the help of the tables:—

1. £27 6s. 4½d. + £308 15s. + £529 6s. 8d. + £34 13s. 9d. + £2000 17s. 8d.

2. £506 18s. 3d. + £27 14s. 3½d. + £9684 3s. 7d. + £12 5s. 3¾d. + £869 14s. 2¼d.

3. £274 8s. 6¼d. + £1200 + £50 4s. 9d. + £783 14s. 5¼d. + £1029 16s. 2½d.

4. £574 15s. 4½d. + £292 18s. 1½d. + £279 16s. 3¾d. + £27 2s. 10¾d. + £69 2s. 6d.

5. 127 cwt. 1 qr. 17 lbs. + 24 cwt. 2 qrs. 27 lbs. + 3 tons 17 cwt. 1 qr. 18 lbs. + 5 tons 6 cwt. 3 qrs. 27 lbs.

6. 32 lbs. 5 oz. 8 dwt. 4 grs. + 3 lbs. 5 dwt. 19 grs. + 4 oz. 17 dwt. 18 grs. + 5 lbs. 6 oz. + 3 oz. 14 dwt.

7. 48 lbs. 13 oz. + 3 cwt. 2 qrs. 9 lbs. + 1 cwt. 3 qrs. 8 lbs. 15 oz. 4 drs. + 3 tons 17 cwt. 6 lbs. + 2 qrs. 9 lbs. 11 oz.

8. 17 miles 3 fur. 19 poles + 28 yds. 2 ft. + 10 in. + 4 miles 3 fur. 8 poles + 7 yds. 2 ft. 9 in.

9. 7 acres 3 roods 9 poles + 2 acres 1 rood 19 poles + 27 acres 3 roods 29 poles 18 square yds. + 52 acres 1 rood 27 poles 12 square yds.

10. 21 qrs. 3 bush. 3 pecks + 5 bush. 7 pecks 2 gals. + 2 bush. 3 pecks 3 gals. 3 qts.

11. Of five packages the first weighs 8 lbs. 3 oz.; the second, 2 qrs. 15 lbs.; the third, 17 lbs. 12 oz. 3 drs.; the fourth as much as the first and third together, and the fifth as much as the first and second together. What is the total weight?

12. From half-past 5 P.M. on the 30th of June to 20 mins. to 11 A.M. on the 5th of September, how much time elapses?

13. Add together 29 acres 3 roods 13 poles, 100 acres 2 roods 1 pole, 85 acres 1 rood 29 poles, 71 acres 3 roods 17 poles, and 12 acres 2 roods 18 poles.

14. One room contains 18 square yds. 3 square ft. 19 in.; a second, 42 square yds. 8 ft. 11 in.; a third, 29 square yds. 5 ft. 100 in.; a fourth, 25 square yds. 2 ft. 81 in.; and a fifth as great an area as all the rest together. What is the total surface?

15. What is the sum of 65 gals. 3 qts., 28 gals. 2 qts., 44 gals. 1 qt., and 83 gals. 2 qts.?

16. If I travel from A to B in 5 hours 7 mins., B to C in 8 hours 12 mins. 3 secs., from C to D in 12 hours 8 mins. 14 secs., and from D to E in 5 hours 16 mins. 25 secs.; in what time can I perform the whole journey from A to E?

17. What is the united length of 6 roads, one measuring 3 leagues 2 miles 7 fur. 30 poles; another, 27 miles 8 yds.; a third, 27 leagues 6 fur. 25 poles; a fourth, 29 miles 3 fur. 18 poles 2 yds.; a fifth, 19 miles 7 fur. 21 poles 2 ft.; and the sixth as much as the first, third, and fifth together?

18. Add together 63 yds. 3 qrs. 1 nail; 74 yds. 2 qrs. 2 nails; 98 yds. 3 qrs. 1 nail; 103 yds. 1 qr. 1 nail; 14 yds. 3 nails; 27 yds. 2 qrs.

19. Find the sum of 3 miles 6 fur. 128 yds. 9 in.; 7 miles 7 fur. 88 yds. 3 in.; 10 miles 3 fur. 25 yds. 6 in.; and 18 miles 5 fur. 205 yds. 11 in.

20. The following sums are owing to a tradesman, £792 10s., £23 18s. £54 6s. 2½d., and £205 10s. 3d. What is the amount?

21. What is the united weight of five parcels, the first of which weighs 17 lbs. 3 oz.; the second, 5 lbs. 13 oz. 12 drs.; the third 11 oz. 9 drs.; the fourth, 4 lbs. 6 drs.; and the fifth, 15 lbs. 15 oz. 15 drs.?

SECTION III.—SUBTRACTION.

36. SUBTRACTION [*] is the process of finding how much greater one number is than another.

The excess is called the Remainder, or the Difference.

Observation.—Subtraction is the opposite of Addition. In addition two numbers are often given, and we are required to find their sum; but in Subtraction, this sum and one of the two numbers are given, and we are required to find the other number.

Example.—"A man has twenty sheep, and seven of them die; how many has he left?" This is a question in Subtraction. It might be expressed in several other ways, *e.g.*, By how many is twenty greater than seven? or, What is the difference between twenty and seven? or, What number of sheep must be added to seven to make twenty?

37. *Sign.* (—,) called *minus*, is the sign of Subtraction.

Thus: $20-7=13$ (read, twenty *minus* seven equals thirteen).

Here twenty is called the *minuend*, seven the *subtrahend*,[†] and thirteen the remainder.

38. AXIOM V.—*We subtract one number from another when we take each of the parts of the first away from the second, in any order whatever.*

Demonstrative Example.—If from £50 I first take away 7, then 6, and then 10, I shall have taken away $7+6+10$, or £23 altogether. The same result would be obtained by taking £20 and then £3, or by taking the parts of the £23 away from the £50 in any other order.

General Formula.—If $a=b+c+d$,

then $x-a=x-b-c-d$.

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[*] *Preliminary Mental Exercise.*—Before any written sums are attempted, some readiness should be acquired in telling the difference between any two numbers.

Select some number as 20, and take away some small number, as 3, from it, and then 3 again as rapidly as possible, 20, 17, 14, 11, 8, 5, 2; or begin with 100, and take away fours successively. For other exercises of this kind see School Arithmetic, p. 13.

[†] From *minuo*, to diminish or lessen, and *subtraho* to subtract. Observe the Latin participles having the termination *and* or *end*; *e.g.*, minuend, that which *has to be* diminished; subtrahend, that which *has to be* subtracted; multiplicand, that which *has to be* multiplied; dividend, that which *has to be* divided.

30. In setting down a sum in Subtraction, it is usual to place the smaller number underneath the greater.

Example.—Find the difference between 342 and 587.

Here 342, the subtrahend, is to be placed beneath 587. The several parts of the one may then be taken from the corresponding parts of the other. Thus,

$$\begin{array}{r c r}
587 & = & 500 + 80 + 7 \\
\underline{342} & = & \underline{300 + 40 + 2} \\
245 & = & 200 + 40 + 5
\end{array}$$

Here it appears that 245 remain after taking 342 away from 587. In such a sum as this it matters little in what order the three separate subtractions are effected; we may either take the hundreds away first, and the rest afterwards, or we may begin with the units.

EXERCISE IX.

☞ (1.) Find the difference between 256 marbles and 132; between 478 and 215; between 3274 and 1162.

(2.) How many must be added to 23 to make 67, to 35 to make 89, to 723 to make 975, to 5321 to make 8765?

40. It often occurs, however, that although the whole subtrahend is less than the minuend, yet some of the parts of the first are greater than the corresponding tens or hundreds of the second. There are two methods of solving such problems; *the method of decomposition* and the *method of equal additions.*

41. *Method of decomposition. Example I.*—"There are 179 cattle in one field, and 342 in another; how many more are there in the one than in the other?"

Here we set down the 342, and underneath it the 179; but although the whole represented in the lower line can be taken from the whole in the upper, yet the 9 units of the lower cannot be subtracted from the 2 above it, nor can the 7 tens be taken from the 4 tens. Here, then, it will be desirable to decompose the minuend into more manageable portions, by carrying one hundred to the tens, and treating it as *ten* tens, and by taking one from the tens place and calling it ten units, thus:—

$$\begin{array}{r c r}
342 & = & 200 + 13 \text{ tens} + 12 \\
\underline{179} & = & \underline{100 + 7 \text{ tens} + 9} \\
163 & = & 100 + 6 \text{ tens} + 3
\end{array}$$

We can now easily subtract the lower line from the upper, as every part is now less than the corresponding part above it.

Example II.—Take 3185 ounces from 7263 ounces.

$$\begin{array}{rcrcrcrcr}
7263 & = & 7000 & + & 100 & + & 15\text{ tens} & + & 13 \\
\underline{3185} & = & \underline{3000} & \underline{+} & \underline{100} & \underline{+} & \underline{8\text{ tens}} & \underline{+} & \underline{5} \\
4078 & = & 4000 & + & 0 & + & 7\text{ tens} & + & 8
\end{array}$$

42. *Observation.*—Here we take 5 not from 3, but from 13, having removed a ten from the 6 tens in the second place; this leaves 8. Then we have to take 8 tens from 5 tens; but this is impossible, so we withdraw 10 tens, or 100, from the 200 in the next place of the minuend. But 8 tens from 15 tens can now be found; they leave 7 tens. 100 has now to be taken from 100, but as the difference here is nothing, we place a cipher to mark that the hundreds place is empty. 3000 taken from 7000 leave 4000, and the answer is thus found to be 4078.

This method, however, though dependent on very simple principles, is not found to be very easy in practice, and is seldom used.

43. *Method of equal additions.*—The ordinary process for working Subtraction is founded on the following truth:—

44. AXIOM VI.—*We do not alter the difference between two unequal numbers if we add an equal number to both.*

Demonstrative Example.—Suppose there are two heaps of stones, one of which contains 50 more stones than the other. It is clear that if 20 more stones be placed on the top of each heap, the one has still 50 more in it than the other. We alter the *magnitude* of each heap by equal additions, but we do not alter the *difference* between them.

General Formula.—If $a - b = x$,
then $(a+c) - (b+c) = x$.

Hence, if in trying to find the difference between any two numbers, it should be convenient to add any numbers to both of them, we are at liberty to do so; for the remainder, which is obtained after making equal additions to the two original numbers, must be the same as would have been obtained before those additions were made.

Thus if we wish to find the difference between 18 and 58, and we choose to add 2 to both before we work the sum, we may do so, for the difference between 18 and 58 must be the same as the difference between 18+2 and 58+2, or between 20 and 60.

When any parts of the subtrahend are greater than the corresponding parts of the minuend, this is the method usually employed.

45. *Example.*—Find the difference between the numbers 20245 and 17386.

$$\begin{array}{r}20245\\17386\\\hline 2859\end{array}$$

We begin this sum by trying to take 6 units from 5: this is impossible. Let us add 10 to the 5; 6 from 15 leave 9. But as 10 were added to the minuend, the same must be added to the subtrahend. There are 8 tens in the next place; let us call this 9 tens, and we shall have added 10 to both the upper and the lower lines. Next try to take 9 tens from 4 tens: this cannot be done, so add [*] 10 tens to the 4; 9 tens from 14 tens leaves 5 tens, which we set down. But having added 10 tens, or 100, to the upper line, we must do the same to the lower, by making the 300 into 400; 400 from 200 cannot be taken, so we add 10 hundreds to the upper line, and say, 400 from 1200 leave 800; this is then to be set down. But to compensate for the addition of 10 hundreds to the upper line, the same, or 1000, must be added to the lower. The 17000 then becomes 18000, which, taken from the 20000, leave 2000, and the answer to the sum is 2859.

46. It should be noticed that by this method we have not actually taken away 17386 from 20245, but that both quantities have received an addition, first, of 10, then of 100, and lastly of 1000, before the subtracting process was finished. The work shown at length is,

		Th.	Hun.	Tens	Units
20245 + 1000 + 100 + 10	or	20	12	14	15
17386 + 1000 + 100 + 10		18	4	9	6
2859		2	8	5	9

The real subtraction effected has not been of 17386 from 20245, but of 17386 + 1110 from 20245 + 1110; that is to say, 18496 has been subtracted from 21355. But according to (44), the difference between these two latter numbers is the same as that between the two original numbers and the desired answer has been obtained, though by an indirect process.

---

[*] The term "borrowing," often applied to this artifice, is not employed here, because it is a very misleading one, and tends to conceal from the learner the real nature of the process of subtraction. It is evident that the principle involved in the ordinary method of working this rule is, that of equal *additions* to minuend and subtrahend, and that nothing whatever is either *borrowed* or paid. It would not be altogether inappropriate to apply the term to the several steps in the process of decomposition described in (41), but here it is only a hindrance to the right understanding of the subject.

RULE FOR SIMPLE SUBTRACTION.

47. Place the less number under the greater, arranging the digits as in Addition. Begin at the right hand, and subtract the units of each kind from the corresponding number above; set down the differences underneath.

Whenever a figure in the lower line is greater than that above it, add 10 the upper, then subtract; and in the next place, on the left, add 1 in the lower line, so that the same sum shall have been added to both.*

Exercise X.

☞ (a) Work the following sums by the method of decomposition, and in the last six sums show at length, as in (41), what is the method employed:—

1. Find the difference between—
79 and 85; 640 and 27; 293 and 1000;
41 and 506; 8097 and 73; 50962 and 1238.

2. Between—
87245 and 37; 2739 and 176; 274 and 8571;
39621 and 2874; 5962 and 4173; 8572 and 4961.

3. 278—37; 400—59; 6281—497;
30000—7294; 4321—897; 6210—891.

(b) Work the following sums by the method of equal additions, and in the last six show, as in (46), what additions have been made to each:—

4. Take Four hundred and ninety-three from a thousand; Seventy-nine from two hundred and thirty-six; Eighty-four from four hundred and fifty; Nine hundred and thirty-seven from two thousand and twenty.

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* In practice, it is well to consider all Subtraction sums as Addition sums in another form; and when calculating, to mention only the number which has to be added to the minuend, and not the subtrahend itself. Thus in the sum—

79234 a good computer will not say, 7 from 14 are 7, 7 from 13 are 6, &c., but
32567 merely looking at the upper numbers he will say, 7 *and* 7 (are 14), 7 *and* 6
46667 (are 13), 6 *and* 6 (are 12), 3 *and* 6 (are 9), 3 *and* 4 (are 7). The words in brackets need not be uttered; it is sufficient to look at the upper row of numbers. It is important in all calculations to use as few words as possible.

5. Take Two thousand six hundred and nineteen from Forty thousand and twelve; Eight hundred and six from a million; Two hundred and forty-five from Seven thousand three hundred.

6. 70968—32975; 3274—596; 30962—1147;
100000—98735; 56934—8973; 20369—8725;
40336—2972; 81572—30961; 48371—8961.

7. Find the difference between—
7209 and 6995; 48372 and 100628; 4718 and 30198;
72386 and 59421; 87243 and 123961; 79632 and 81405;
2071 and 500000; 5098 and 37.

8. The greater of two numbers is 1004, and their difference 49, what is their sum?

9. Take the sum of 798 and 6251 from the sum of 72835 and 6109. Also, 587+6403 from 5962+8471+9274.

10. If from a sum of £92384 I pay £625 to one person, £804 to another, £2096 to a third, and £1527 to a fourth, how much will be left?

11. How long is it since the year 543 A.D? since 62 A.D.? since 1057 A.D.? since 1666 A.D.? since 185 A.D.?

12. If the sum of two numbers be 968547, and the less be 209682, what is the greater, and what is their difference?

13. If one person was born in 1810, another in 1795, a third in 1839, and a fourth in 1842, how old would each be in 1854?

14. If a man owes £791 to one person, £683 to another, £4627 to another, and £1629 to a fourth; and if to him one person owes £2086, another £56, and another £5905, how do his affairs stand?

15. Add the sum of 9546 and 3285 to their difference.

16. What number added to the sum of 596 and 3024 will give the sum of 5096, 2837, and 2462?

17. One ship contains 7928 pounds of merchandise, a second contains 39254, and a third 20638. What are the differences between the contents (*a*) of the first and second, (*b*) the first and third, and (*c*) the second and third?

18. Add together 17, 19, 53, and 40, and subtract the sum from that of 19, 64. 83, and 106.

19. (18+6+209+537) — (628+31+19).

20. (1786 — 9 — 12) — (235+672 — 751).

21. The number of the male population in Great Britain in 1871 was 11,040,403, and of the female, 11,663,705; what was the whole population of the island, and what was the excess of females over males?

METHODS OF PROVING SUBTRACTION.

48. I. From (36) it appears that the remainder, or answer, added to the subtrahend, ought to make up the minuend. The simplest method of verifying any answer is, therefore, to add it to the lower line in the sum. If the result corresponds to the upper, the answer must be right.

Thus,
5096384 = minuend.
2739725 = subtrahend.
2356659 = remainder.
5096384 = remainder + subtrahend.

49. II. Cast out the nines from the digits of the minuend, and set down the excess by the side of the line; do the same with the figures of the subtrahend. If the differenee between the two numbers thus found is the same as the excess after casting out the nines from the last line or remainder, the answer is probably right. (See 175.)

Example.—		
	7638542	8
	1975768	7
	5662774	1

SECTION IV.—COMPOUND SUBTRACTION.

50. *Method of decomposition.*—Take £5 9s. 6d. from £8 3s. 2d. Suppose that the greater sum of money is in the form of 8 sovereigns, 3 shilling-pieces, and 2 penny-pieces, and that we have to pay away £5 9s. 6d. Here, although we have enough money and to spare, yet we have not enough pence, nor enough shillings, while the money is in its present form. Such a payment will of course be made by getting change for one of the shillings, and turning it into 12 pence; and by changing one of the sovereigns, in like manner, into 20 shillings. Thus,

£	s.	d.		£	s.	d.
8	3	2	will take the form of	7	22	14
			From which, if we take away	5	9	6
			There will remain . .	2	13	8

This method will of course be equally suitable to hours and minutes, or to gallons and quarts, or to miles and yards, or any form of concrete quantity. When the Tables are known, the learner will easily be able to work the questions in the following exercise.

EXERCISE XI.

☞ Decompose the following sums as in the examples :—

	tons.	cwt.	qrs.	lbs.	oz.		tons.	cwt.	qrs.	lbs.	oz.
Example I.—	25	16	3	4	2	=	24	35	6	31	18

	miles.	fur.	poles.	yds.	ft.	in.		miles.	fur.	poles.	yds.	ft.	in.
Example II.—	12	2	13	2	2	7	=	11	9	52	6½	4	19

(1.) 5 years 217 days 17 hours 54 mins. ; 27 weeks 3 days 4 hours.

(2.) 5 tons 3 cwt. 2 qrs. 10 lbs. ; 47 lbs. 7 oz. 6 dwt. 5 grs.

(3.) 276 qrs. 3 bush. 2 pecks 1 gal. ; 18 hrs. 2 mins. 17 secs.

(4.) 11 yds. 2 ft. 8 in. ; 2 miles 3 fur. 7 poles; 5 miles 8 yds.

51. The *Method of equal additions* is, however, that by which such sums are usually worked.

Example I.—Take £79 17s. 3d. from £101 3s. 1d.

£	s.	d.		£	£	s.	d.
101	3	1	=	10 tens	1 + 10	3 + 20	1 + 12
79	17	3		7 tens + 1 ten	9 + 1	17 + 1	3
21	5	10		2 tens	1	5	10

£	s.	d.
100 + 11	23	13
80 + 10	18	3
20 + 1	5	10

We first try to take the pence of the subtrahend from those of the minuend (3 from 1) : this cannot be done, so we add 12 pence to the upper line : 3 pence from 13 pence leaves 10 pence. But having added 1 shilling to the upper line, we must also add one to the lower, so we next say, 18 shillings from 3 ; but this is impossible, so we add £1 or 20 shillings to the upper line, and say 18 from 23 leaves 5 ; then having increased the upper line by the addition of £1 we must do the same to the lower ; afterwards we proceed as in Simple Subtraction. It will be seen that £10 + £1 + 1s. have been added to both lines.

52. *Observation.*—The additions thus made to both the quantities, though equal, are different in form. To the upper line, the addition is made in the form of 12 pence, but to the lower in the form of 1 shilling; to the upper in the form of 20 shillings, but to the lower in the form of 1 pound. But the terms of Axiom VI. are still complied with, because we do in fact add equal amounts to both.

	miles.	fur.	poles.	yds.	ft.		miles.	fur.	poles.	yds.	ft.
Example II.—	3	5	23	2	2	=	3	13	63	7½	2
	2	7	24	4	1		3	8	25	4	1
	0	5	38	3½	1		0	5	38	3½	

1 mile 1 fur. 1 pole have been added to both these quantities.

RULE FOR COMPOUND SUBTRACTION.

53. Place the less number under the greater, and arrange them as in Compound Addition. Begin at the right hand, and take away the parts of the less from the corresponding parts of the greater, setting down the differences underneath. If a number in the subtrahend be greater than that above it, add to the upper line as many of that quantity as make one of the next higher denomination; then add one of the same denomination to the subtrahend. Proceed in this manner until the whole subtraction is effected.

EXERCISE XII.

☞ (a) Solve the following questions; the first six by the method of decomposition, and all by the method of equal additions.

(b) State, in the case of the last ten, what quantities have been added to each. (See Example II.)

1. Find the difference—

Between £79 16s. 3d. and £28 7s. 2½d.; £125 and £16 2s. 9d.;

2. £826 13s. 4d. and £123 7s. 3d.; £246 13s. 11d. and £298 3s. 6d.

3. Take £297 16s. 3d. from £1000.

4. £5287 16s. 5½d.—479 15s. 10d.; £500—£279 16s. 3d.

5. £276 13s. 3d.—£49 17s. 8½d.; £4056 10s. — £274 16s. 3¼d.

6. £2741 5s. 6d. — £3 1s. 8d.; £3004 — £298 16s. 3½d.

7. £19862 14s. 5½d.—£7934 15s. 6¼d.; £100 — £89 17s. 5d.

8. £72385 10s. 4¾d. — £6985 12s. 7¾d. — £2728 12s. 5¼d.

9. 49 weeks — 27 weeks 3 days 5 hours.

10. 23 miles 3 fur. 8 poles — 7 miles 3 fur. 4 poles 2 yds.

11. 7 leagues 2 miles — 9 miles 3 fur. 12 yds.

12. Take from £1500 the following sums—£158 3s. 7¼d., £62 9s. 7½d., and £54 3s. 4d. How much is left?

13. Find the difference between 1 ton and 7 cwt. 3 qrs. 14 lbs. and between 17 lbs. 13 oz. and 2 qrs. 9 lbs.

14. Take 8 cwt. 1 qr. 17 lbs. from 12 cwt. 3 qrs. 5 lbs.; 9 lbs. 10 oz. from 4 lbs. 3 oz. 12 drs.

15. By how much does 18 miles exceed 12 miles 3 fur. 17 poles?

16. If I owe £79 16s. 4d., £18 3s. 2d., £27 15s. 3d½., and possess

£600, while A owes me £15 9s. 6¼d., and B £23 18s. 2¼d., what balance will remain when all my debts have been received and paid?

17. Subtract 72 tons 15 cwt. 3 qrs. from 100 tons 9 cwt. 1 qr. 12 lbs.

18. Subtract 25 acres 3 roods 4 poles from 105 acres.

19. Subtract 3 leagues 2 miles 7 fur. from 7 leagues 1 mile 9 poles.

20. Subtract 56 yds. 2 qrs. 1 nail from 75 yds. 1 qr.

21. Subtract 5 bush. 3 pecks 2 qts. from 7 qrs.

22. If at 7 mins. past 2 A.M. on April 3rd, 1827, a man was 36 years 19 days and 5 hours old, when was he born?

23. If I possess £764 2s. 8d., and B has £29 3s. 5¼d. less than I, what sum have we together?

24. 27 cwt. 3 qrs. 12 lbs. 7 oz.—19 cwt. 2 qrs. 18 lbs. 7 drs.
28 tons 15 cwt. 2 qrs. 9 lbs.—11 tons 19 cwt. 3 qrs. 7 oz.

25. 15 st. 4 lbs. 11 oz. 3 drs.—25 lbs. 3 oz. 6 drs.
27 st. 7 lbs. 6 oz. 5 drs.—13 st. 9 lbs. 12 drs.

26. 50 miles 3 fur. 17 yds.—28 miles 3 fur. 23 poles 2 yds.
19 yds. 2 ft. 11 in.—12 yds. 14 in.

27. 25 acres 3 roods 28 poles—6 acres 2 roods 39 poles.
37 acres 3 roods 29 poles 27 sq. yds.—4 acres 2 roods 11 poles 29½ sq. yds.

28. From a meadow of 1 acre 2 roods, 3 roods 4 poles are cut off for a kitchen-garden; how much remains?

29. Two boys are respectively 11 years 5 weeks and 4 days, and 8 years 17 weeks and 1 day old; what is the difference between their united ages and that of their father, who is 50 years old?

30. What is the difference in the length of the solar and sidereal years; the former being 365 days 5 hours 48 mins. and 57 secs., and the latter, 365 days 6 hours 9 mins. and 11 secs.?

31. What is the difference in latitude between Lizard Point, the most southern part of Great Britain, 49° 58′ N., and Cape Wrath, the most northern, 58° 38′ N.?

32. How much is an English sovereign worth more than the sum of a Russian silver rouble, a French franc, a United States dollar, and a Portuguese milrea? (For the value of these coins, see Appendix.)

33. Find the difference between the sum of 17 lbs. 6 oz., 29 lbs. 3 oz. 4 drs., 18 lbs. 11 drs., and 2 lbs. 9 oz. 10 drs., and the sum of 3 qrs. 5 lbs. and half a hundredweight.

34. What is the difference in length between two roads, of 97 miles 3 fur. and of 18 miles 23 poles 17 yds. in length respectively, and one 120 miles long?

** SECTION V.—ADDITION AND SUBTRACTION.

Sign. ( ). Whenever numbers are enclosed in brackets, the sign before them applies to the whole quantity within the brackets.

$\therefore$ means *therefore.*

e. g., $16-(5+4)=16-9=7.$

$16+(5-4)=16+1=17.$

54. *If there be three numbers, such that the first is as many more or less than the second, as the second is more or less than the third, the sum of the first and third will equal the second added to itself.*

Demonstrative Example.—12, 7, 2. Because 7 is as much less than 12 as it is more than 2, $\therefore$ $12+2=7+7$.

For, by (25), if what is taken away from one of the sevens be added to the other the sum will remain the same.

If b be as many less than a as it is more than c;

or if b be as many more than a as it is less than c,

then in either case $a+c=b+b=2b$.

EXERCISE XIII.

☞ (a) Find two numbers whose sum will equal—

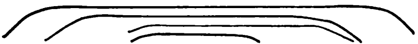
$8+8$; $12+12$; $53+53$; $19+19$.

$100+100$; $74+74$; $327+327$; $568+568$.

(b) Place in each vacant space a third number which, if added to the first, will equal the second added to itself;

27, 18, ( ); 47, 54, ( );
3, 17, ( ); 29, 16, ( );
18, 100, ( ); 293, 500, ( );
409, 400, ( ); x, $x-3$, ( );
$a-3$, a, ( ); $a+5$, a, ( ).

Corollary I.—In any series of numbers so arranged that there is the same difference between every two adjacent numbers, the sum of any two which are equally distant from a given number equals that number added to itself.



Example.—2 5 8 11 14 17 20 23 26

Observe, these numbers are so arranged that between each and the next there is a difference of three.

5, for example, is as many less than 14 as 23 is more than 14, hence $5+23=14+14$. So any number added to itself gives the same amount as the sum of any two equally distant from it, *e. g.*—

$5+11=8+8$, $17+23=20+20$, $23+11=17+17$, &c.

Corollary II.[*]—In every such series of numbers the sum of the two extreme terms equals the sum of any two terms at equal distances from the extremities, and the whole series is made up of a number of pairs, each pair having the same sum as the two extremes.

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[*] This corollary forms the basis of an important rule in Arithmetical Progression. See Sec. 504.

EXERCISE XIV.

☞ Make 12 series of 15 numbers each, and show how each illustrates this principle.

55. *If there be four numbers, of which the second is as many more or less than the first as the third is less or more than the fourth, the sum of the first and fourth will equal that of the second and third.*

Demonstrative Example.—5, 9, 100, 104. By (25), if the two numbers, 9 and 100, are taken, and if what is removed from the 9 to make 5 be carried to the 100 to make 104, the sums will be the same; *i. e.*, $9 + 100 = (9 - 4) + (100 + 4)$; or $5 + 104 = 9 + 100$.

If of the quantities a, b, c, and d, a be as many more than b, as d is less than c; or if b be as many more than a, as is less than d, then $a + d = b + c$.

EXERCISE XV.

☞ Place a fourth number in each of the following series, so that if added to the first it will equal the sum of the second and third.

9, 7, 20, ( ); 20, 15, 8, ( ); 3, 12, 70, ( ); 29, 5, 64, ( ).
$a+3, a, b$, ( ); $x-5, x, y$, ( ).
$a+7, a, x$, ( ); $m+x, m, n$, ( ).
Suppose $p = q - 6$;
$q, p, 15$ ( ); $p, q, 3$ ( ); p, q, m, ( ).

Questions on Addition and Subtraction.

What is addition? Give an example. Quote one of the principles on which the rule for working it depends. What part of the rule is shown to be necessary by this principle? What sort of numbers are they which ought not to be added together? Why? Repeat the axiom on this subject. What part of the rule does this explain?

Why are numbers placed in columns for Addition? What is carrying? What axiom is assumed when we carry a figure to the next column? What is it to prove a sum? How will you prove an Addition sum? State the Rule for Simple Addition. For Compound. Explain the terms subtrahend, difference, sum, and minuend. Describe the method of equal additions. Of decomposition. State each of the axioms assumed in Subtraction, and give an example of each.

Why is it better to commence a sum on the right hand? What equal additions are always made in a Subtraction sum? Why? How may a Subtraction sum be proved? Give the reason. Give the Rule for Compound Subtraction.

State what general truth is illustrated by each of the following equations:—

1. If $w = x + y + z$, then $a + w = a + x + y + z$.
2. and $a - w = a - x - y - z$.
3. $a - c = (a + x) - (c + x)$.
4. $20 + 30 = 25 + 25$, and $14 + 94 = 17 + 91$.
5. $(x - y) + (x + y) = x + x$.
6. $(x - y) + (a + y) = x + a$.

MULTIPLICATION AND DIVISION.

SECTION I.—SIMPLE MULTIPLICATION.

56. To multiply a number is to repeat it, or add it to itself, a certain number of times.

The number thus repeated is called the *Multiplicand;*[*] the number of repetitions is called the *Multiplier*, and the answer is called the *Product.* The multiplier is an *abstract* number, for it expresses the number of *times* that the multiplicand is repeated. It can never be concrete.

Observation.—In certain cases it is convenient to call both multiplicand and multiplier *factors*, they being the two numbers which make (*facio*) or produce the answer.

57. To multiply is to find a number which is as many times greater than the multiplicand as the multiplier is greater than unity.

58. *Sign.* × is the sign of multiplication. Thus 4 × 6 (read 4 *times* 6, or 4 *into* 6) = 24.

$a \times b = a$ multiplied by b, or a taken b times. This is generally written ab.

Example.—"There are 5 bags of money and 7 sovereigns in each; how many are there in all?" Here 7 sovereigns require to be multiplied or repeated, and are called the Multiplicand; the figure 5 shows how many times the 7 sovereigns are to be taken, and is called the Multiplier; and the answer, 35 sovereigns, is the product, which is an amount as many times greater than the multiplicand (seven) as the multiplier (five) is more than unity.

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[*] The word Multiplication literally means manifolding; it is derived from *multus*, many, and *plico*, to fold. Multipli*cand* means that number which *is to be* multiplied. Multiplier is that which shows how many times this multiplicand is to be taken. Product (*pro-duco*), is the result which is *produced* by the process.

Preliminary Mental Exercise.—As in Addition and Subtraction, it is necessary here to make ourselves familiar with the relations between a few simple numbers, such as those given in School Arithmetic, p. 17. For the same purpose the pupil should also make for himself a table, like that in the text, showing the result of a series of additions to each of the first 12 numbers, and afterwards learn it by heart.

59. In the following Table it will be seen that the first 12 numbers are arranged in a horizontal line at the top, beginning with the number 1. In the second line the number 2 is taken, and a series of twos are added to it in order, so that three twos stand under the number 3, and five twos under the number 5, &c. In like manner a set of equal additions is made to each of the first twelve numbers in successive horizontal lines. Hence, to find 9 times 6, I look along the sixth horizontal line, until I find the number exactly under the 9, which is 54; or the sum of 9 additions of 6. The product of any other two numbers between 1 and 12 can be found in the same way.

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

The numbers which form the diagonal of this figure, and which are enclosed in darker lines, are called the *squares* of the numbers at the top of the columns to which they belong. It will be observed that the numbers on each side of this line are the same, but in a different order.

60. Multiplication is abbreviated Addition, but it will only enable us to perform those Addition sums in which it is required to add together the *same* numbers.

Example.—If it be required to find what 4 times 57 will make, the sum may be stated as in the margin. In the units column, all the numbers to be added together are alike; but it is unnecessary to make three different additions to the 7, if we know that 4 sevens make 28. In like manner, we need not make several mental efforts in order to add the 4 fives together, if we have learnt from the table that four times five make 20. But if all the lines of figures in the sum had not been alike, the Multiplication Table would not have helped us: we should have been obliged to obtain the answer by ordinary addition.

$$\begin{array}{r}57\\57\\57\\57\\\hline 228\end{array}$$

61. Multiplication, like Addition and Subtraction, is a distributive operation. When it is performed on each of the parts separately, it is performed upon the whole.

62. AXIOM VII.—*We multiply one number by another when we multiply each of the parts of the first by the second and add the several products together.*

Demonstrative Example.—To multiply seven by five, we may take the parts of seven and multiply each by 5, thus:—

Because $7 = 4 + 2 + 1$,

therefore $5 \times 7 = (5 \times 4) + (5 \times 2) + (5 \times 1)$,

or $20 + 10 + 5 = 35$.

General Formula.—If $a = b + c + d$,

then $na = nb + nc + nd$.

To multiply a number, as 7962, which is composed of four parts, viz., 7000 + 900 + 60 + 2 by any number, as 8; we may multiply each of these parts in order, thus:—

$$\begin{array}{c|c|c|c} 7 & 9 & 6 & 2 \\ & & & 8 \\ \hline 56 & 72 & 48 & 16 \end{array} = 63696$$

the answer thus obtained, 56 thousands + 72 hundreds + 48 tens + 16 is the true one, and only needs to be corrected, as to its form, by *carrying* in accordance with the rule given in (25). It then becomes 63696.

63. CASE I.—TO MULTIPLY BY ANY NUMBER LESS THAN THIRTEEN—

RULE.—Place the multiplier under the unit of the multiplicand; find the product of the multiplier and number above it; set down the units digit of the answer, carrying the tens into the next place, as in Addition. Then find the product of the multiplier and the figure in the tens place; set down the odd tens, carry the hundreds, and proceed with each figure in the same manner.

EXERCISE XVI.

☞ Find the product of the following numbers :—

1. 283 and 7; 50629 and 5; 2964 and 4; 3274 and 8.

2 20847 and 9; 618 and 11; 2934107 and 12.

3. 729 × 6; 84237 × 7; 5864 × 9; 712348 × 9; 58574 × 2.

4. 526387 × 7; 201306 × 5; 4158692 × 7; 723486 × 7.

5. 424374 × 8; 386 × 9; 84729 × 6; 3964187 × 5.

6. 413826 × 6; 702987 × 9; 546208 × 8; 312628 × 7.

7. Whati s the difference between seven dozen and five, and twenty-six dozen and three?

8. If each child in a school of 135 had five marbles, and each in a school of 73 had eight, how many would there be in all?

9. In five chests there are two thousand oranges; one contains thirty-two dozen, and three others contain thirty-five dozen and six in each; how many are in the fifth?

10. There is a monthly publication containing 36 pages; how many pages will there be in eight half-yearly volumes of it when bound?

11. If a town contains 2748 houses, and on an average three grown persons and four children in each, what is its population?

12. What is the difference between the sum of forty-seven dozen and eight score, and the product of eighty-six and eleven?

13. What is the number of children in a large school of two departments, having in the one seven classes containing on an average 28 pupils each, and in the other nine classes with 26 pupils each?

64. AXIOM VIII.—*Multiplying one number by a second gives the same result as multiplying the second by the first.*[*]

Demonstrative Example.—3 times 8 is the same as 8 times 3.

o o o o o o o o
o o o o o o o o
o o o o o o o o

The whole of these ciphers may either be considered as three rows of eight each, or as eight rows of three each. By reference to the Multiplication Table it will be seen that the number 24 occurs twice, once in the third column, on the line of eights (3 times 8), and once in the eighth column, in the series of threes (8 times 3).

General Formula.—$a \times b = b \times a$.

65. *Corollary I.—The product of any number of factors is the same, in whatever order they are multiplied.*

Example.—$3 \times 7 \times 6 = 6 \times 7 \times 3$, or $7 \times 6 \times 3$.

General Formula.—$a\ b\ c\ d = b\ d\ c\ a = d\ b\ a\ e$.

66. *Corollary II.—We multiply by the whole of a number when we multiply successively by its factors.*

Demonstrative Example.—$1 \times 15 = 15$.

$$\text{Also, } 1 \times 3 \times 5 = 15,$$

$$\therefore \quad 1 \times 15 = 1 \times 3 \times 5.$$

But what is true of a unit is true of 2 units or 3 units or of n units; that is, of any number. Therefore we multiply one number by the whole of another when we multiply by its factors.

General Formula.—If $a \times b = x$, then $nx = (na)\ b$.

---

[*] This principle, which appears almost self-evident, and is assumed without demonstration in ordinary Arithmetic, is made the subject of one of the propositions in the seventh book of Euclid's Elements. The following is the method of proof:—

Prop. XVI. If one number, A, be multiplied by B, the product is the same as if B were multiplied by A, *i. e.*, AB = BA.

Suppose B × A gives a certain result, x, then, according to the definition of Multiplication, x contains B as many times as A contains 1.

$$\therefore \text{(I.)}\ 1 : A :: B : x.$$

Similarly, if A × B gives a certain result, y, then, for the same reason,

$$\text{(II.)}\ 1 : B :: A : y.$$

But by a former proposition, "If the first be to the second as the third is to the fourth, then the first is to the third as the second is to the fourth;" hence the last proposition may take the form, $1 : A :: B : y$, but by (I.) $1 : A :: B : x$. Hence, B is contained as often in x as in y, and $x = y$, or BA = AB....—*Q E.D.*

67. CASE II.—WHEN THE MULTIPLIER IS GREATER THAN TWELVE, AND ITS FACTORS ARE KNOWN—

RULE.—Multiply by one factor, and that product by another factor, and so on till all the factors have been employed.

Example.—Multiply 3247 by 56. Here $56 = 7 \times 8$.

$$\begin{array}{r l} 3247 & \\ 7 & \\ \hline 22729 & =3247\times 7 \\ 8 & \\ \hline 181832 & =3247\times 7\times 8=3247\times 56. \end{array}$$

EXERCISE XVII.

1. Multiply 23972 by 28 ; 564 by 18 ; 2740 by 72.

2. 20974 by 144 ; 87412 by 35 ; 4693 by 49 ; 847209 by 63.

3. 962187 by 24 ; 30962 by 55 ; 8291 by 45 ; 357 by 240.

4. If a train travels at the rate of 28 miles an hour, how far will it go in 37 hours?

5. In a town there are 734 houses; 345 of them contain, on an average, 11 persons each, and the rest 15 each. How many persons reside in the town?

6. A man walks up and down a flight of 57 steps 25 times a day; how many steps does he take upon it?

7. If there are 24 sheets in a quire, and 20 quires in a ream, how many sheets are in 359 reams?

8. How many nuts are there in 56 bags containing 19648 each?

68. It follows, from the principle of our notation (15), that to place a cipher (0) on the right of a line of figures has the effect of multiplying the whole by ten. For each figure on being removed one place to the left means ten times more than before. In the same way, placing two ciphers to the right of a number multiplies it by 100; three ciphers, by 1000, and so on with all the higher denominations.

Example— $496 \times 10 = 4960.$

Placing a cipher after this number alters the 6 into 6 tens; the 9 tens into 9 tens of tens, or 9 hundreds; and the 4 hundreds into 4000, or 10 times 400. Two ciphers would have had the effect of multiplying by 100; three of multiplying by 1000; four by 10,000, &c.

69. CASE III.—WHEN ANY NUMBER OF CIPHERS IS TO THE RIGHT OF A MULTIPLIER—

RULE.—Multiply only by the significant figures, and add as many ciphers to the answer as there are in the multiplier.

Example.—Multiply 79834 by 400.

$$\begin{array}{ll} \text{First multiply it by 4.} & \begin{array}{r} 79834 \\ 4 \\ \hline 319336 = 4 \times 79834 \\ \hline \end{array} \\ \text{Add 2 ciphers.} & 31933600 = 4 \times 100 \times 79834. \end{array}$$

70. *Observation.*—This answer has been gained by two steps. Because 400 = 4 × 100, we first multiplied by 4; but in adding two ciphers to this product, every figure in it becomes 100 times what it meant before, and the whole has therefore been increased by 4 × 100, or 400 times. In practice it is not necessary to write down two lines as in the examples, as the ciphers are set down on the right before the multiplication begins.

EXERCISE XVIII.

1. Multiply 8746 by 30; 2974 by 200; 8274 by 3000.
2. 56847 by 700; 3178 by 7000; 296845 by 90000.
3. 21869 by 80000; 314721 by 50000; 23896 by 60000.
4. 58471 by 400; 9874 by 80000.

We have now seen how to multiply by any number greater than twelve, when that number is capable of being resolved either into simple factors, or into two factors, one of which is 10, 100, 1000, &c. But many numbers—such, for example, as 37, 146—cannot be treated by either of these methods. The ordinary process of working such questions depends on the following truth :—

71. *We multiply one number by another, when we multiply it by each of the parts of the other in succession, and add the products together.*

Demonstrative Example.—Because $7 = 3 + 4$,
therefore $29 \times 7 = (29 \times 3) + (29 \times 4)$.

So to find the product of 238 and 42, we may take 238 × 40 and add to it 238 × 2, and the sum of these products will be the answer.

General Formula.—If $a = b + c$, $na = nb + nc$.

This principle, taken in connexion with that enunciated in (62), gives us the following general truth.

72. ***We multiply one number by another, when we multiply each of the parts of the one by each of the parts of the other, and add their products together.***

Demonstrative Example.—$25 \times 78 = (20 + 5) \times (70 + 8)$.

And $(20 \times 70) + (5 \times 70) + (20 \times 8) + (5 \times 8) = 25 \times 78$.

General Formula.—If $a = b + c$, and $x = y + z$,

then $ax = by + bz + cy + cz$.

Example II.—Multiply 723 by 364.

Here the multiplier consists of three parts, 300, 60, and 4; but if 723 is multiplied by each of these parts the sum of the products will give the required answer.

$$
\begin{array}{r l}
723 & \\
364 & \\
\hline
2892 & = 723 \times 4 \\
43380 & = 723 \times 6 \times 10 \\
216900 & = 723 \times 3 \times 100 \\
\hline
263172 & = 723 \times (4 + 60 + 300) = 723 \times 364.
\end{array}
$$

We first multiply the parts of the multiplicand (62) by 4, and find the answer to be 2892; we have next to multiply by 60, but since $60 = 6 \times 10$, we may, by (69), multiply by 6 and place a cipher after it, in this way 723×60 is found to be 43380. It now remains to multiply 723 by 300. By (69) this may be done by multiplying by 3 and placing two ciphers to the right of the product, hence $723 \times 300 = 216900$. The sum of these three products is the product of 723 and 364; *i. e.*,—

$(700 + 20 + 3) \times (300 + 60 + 4)$ has been found by multiplying each of the parts of the one factor into each of the parts of the other.

73. But in this operation more figures have been employed than are necessary. The cipher in the second line, and the two ciphers in the third, are of no use except to show the value of the figures that stand before them. This value is secured in practice by placing the first figure obtained in each line under the multiplier which obtained it; *e. g.*, in multiplying by the unit, the first figure will be in the units place; in multiplying by the tens, the first result will be placed under the tens; by the hundreds, under the hundreds, and so on.

RULE FOR LONG MULTIPLICATION.

74. Write the multiplier under the multiplicand. Multiply first by the unit, setting down the answer as in the simple rule ; next multiply by the tens, placing the unit of this answer in the tens place. Place, in like manner, the first figure of the hundreds product in the hundreds place, of the thousands product in the thousands place, &c., and add them up, to find the total product.

75. Sometimes one or more ciphers occur in a multiplier. Whenever this happens, remember that units multiplied by units give units as their product ; by tens, give tens ; by hundreds, give hundreds ; &c. It is necessary, therefore, to think of the value of each portion of the multiplier as it is used, to remember that the first result obtained has the same value, and to place it accordingly.

Examples.—Multiply 2798 by 204 ; and 63157 by 3005.

I.	II.
2798	63157
204	3005
$11192 = 2798 \times 4$	$315785 = 63157 \times 5$
$5596 = 2798 \times 200$	$189471 = 63157 \times 3000$
$570792 = 2798 \times (200 + 4).$	$189786785 = 63157 \times (3000 + 5)$

76. In (I.) it was necessary to observe that the 2 in the multiplier meant 2 hundreds, and that the first product (twice 8) therefore meant 16 hundreds, and must stand in the hundreds place. In (II.) the 3 of the multiplier meaning 3 thousands, 3 times 7 gives, as a product, 21 thousands, and the 1 therefore stands in the thousands place, or three figures to the left of the unit.

77. *Observation.*—Although in the process of Multiplication it is easy to see the necessity for beginning the operation on the right, in order that numbers of a certain degree may the more conveniently be transferred from the lower place, there is no reason, apart from the desirableness of uniformity, why the order of the partial multiplications should not be inverted or altered in any way we please. For example :—

$$\begin{array}{rl}
37286 & \\
264 & \\
\hline
74572 & =37286\times200\\
223716 & =37286\times60\\
149144 & =37286\times4\\
\hline
9843504 & =37286\times264
\end{array}$$

Here the first partial product obtained was that of the multiplicand by 200, which was therefore put in the hundreds place; the second part of the multiplier was 60, and the first figure of the product took the tens place. The third product consists of a number of units, and stands in the same relation to the rest of the sum as if it had been placed at the top in the usual way.

Exercise XIX.

1. Find the product of 279 and 14; of 8793 and 401.
2. Of 17986 and 3709; of 274, 302, and 87; of 53 and 6948.
3. Of 77 and 810325; of 968 and 324; of 175 and 809625.
4. Multiply 796284 by 37; 829741 by 59.
5. 304156 by 168; 7428327 by 539; 920685 by 7098.
6. 27468 by 3974; 814729 by 257; 638041 by 724.
7. 57412 by 387; 6541 by 729; 858604 by 23.
8. 5684 × 301 × 29; 407 × 18 × 5; 5096 × 15 × 13.
9. 70271 × 21 × 8; 52965 × 4 × 17.
10. 378596 × 4 × 13; 729 × 8 × 61; 3190865 × 5 × 7.
11. 418327 × 6 × 9; 7184 × 6 × 2.
12. If I have 3 purses containing £17 each; 5 containing £29 each, and 12 £5 notes, how much money have I?
13. What is the difference between the product of 18, 19, and 35, and that of 24, 17, and 12?
14. Find the product of two numbers, the greater of which is 1694, and the difference 189.
15. How many pens are there in 697 gross?
16. If a master employs 73 workmen, each of whom receives 34 shillings per week, how many shillings does he pay away weekly?
17. Trajan's bridge, over the Danube, is said to have had 20 piers to support the arches. Every pier was 60 feet broad, and they were 170 feet asunder; what was the width of the river in that place?
18. If of 20,000 shells used in war, 3,648 are 36 pounders, 11,275 are 24 pounders, and the rest 18 pounders, what is the total weight (in pounds) of the whole?
19. The hammer of a clock strikes 156 times a day; how many times does it strike in 3 years of 365 days each?

SECTION II.—COMPOUND MULTIPLICATION.

RULE.

78. When the multiplier is not greater than 12 begin with the numbers of the lowest name, multiply each in turn, and carry as in Addition.

But when the multiplier is greater than 12 resolve it into its factors, and multiply by each in succession. If no factors can be found which will exactly produce the multiplier, take the nearest, then multiply the top line by the difference between this product and the multiplier, and add the new product to the rest.

Example I.—Multiply £63 17s. 10½d. by 274. Here we observe that 270 is made up thus: 9 × 3 × 10 = 270.

The sum of money must therefore be multiplied first by these numbers in succession, and 4 times the original sum afterwards added to the product.

£	s.	d.	
63	17	10½	× 274, or $(\overline{9 \times 3 \times 10} + 4)$
		9	
575	0	10½	= 63 17 10½ × 9
		3	
1725	2	7½	= 63 17 10½ × 9 × 3, or 27
		10	
17251	6	3	= 63 17 10½ × 9 × 3 × 10, or 270
255	11	6	= 63 17 10½ × 4
17506	17	9	= 63 17 10½ × 274

Example II.—Multiply 17 cwt. 3 qrs. 20 lbs. 7 oz. by 97.

tons.	cwt.	qrs.	lbs.	oz.	
0	17	3	20	7	× 97, or $(\overline{12 \times 8} + 1)$
				12	
10	5	0	21	4	= 17 cwt. 3 qrs. 20 lbs. 7 oz. × 12
				8	
82	1	2	4	0	= 17 cwt. 3 qrs. 20 lbs. 7 oz. × 12 × 8, or 96
0	17	3	20	7	
82	19	1	24	7	= 17 cwt. 3 qrs. 20 lbs. 7 oz. × $(\overline{12 \times 8} + 1)$ or 97.

79. *Observation.*—The method of Long Multiplication employed in (74) depends on the fact that each figure of the multiplicand, if removed one place to the left, would mean ten times as much. It is therefore not suitable for lines consisting of pounds, shillings, and pence, or any other concrete numbers. For example, the number 263 would be multiplied by 10 if a cipher were added. It would then be 2630. But 2 miles 6 fur. 3 poles could not be treated in this way, but would require to be separately multiplied. The method of Long Multiplication, which depends on the resolution of the multiplier into its factors (67), is, therefore, to be used for compound or concrete numbers.

EXERCISE XX.

☞ (a) Resolve the following numbers into a form suited for compound multipliers.

Example.— $822 = (9 \times 9 \times 10) + 12.$
$726 = (12 \times 6 \times 10) + 6.$

563, 87, 39, 403, 1227, 645, 91,
832, 407, 86, 93, 29, 5325.

(b) In working the following sums, set down, in the first six, the separate value of each line, as in examples I. and II.

1. Multiply £7 3s. 8d. by 5; £14 15s. 6d. by 4; £2 9s. 3¾d. by 7.
2. £17 8s. 3d. by 11; £34 17s. 8½d. by 9; £235 6s. 3¼d. by 6.
3. £2745 18s. 11½d. by 8; £695 14s. 2½d. by 15; £27340 18s. 3d. by 24.
4. £5629 7s. 2½d. by 95; £37205 by 78; £46027 4s. 2¼d. by 73.
5. £8954 7s. 4½d. by 179; £7219 12s. 5d. by 387.
6. 13 acres 3 roods 25 poles by 7; 4 tons 17 cwt. 19 lbs. by 8. 4 miles 7 fur. 8 poles by 13.
7. 17 miles 7 fur. 29 poles 3 yds. by 27; 17 lbs. 10 oz. 15 dwt. 4 grs. by 19; 5 lbs. 7 oz. 2 scruples by 46.
8. 3 cwt. 2 qrs. 7 lbs × 29; 7 tons 2 cwt. 18 lbs. × 35; 14 cwt. 2 qrs. 27 lbs. × 138.
9. 17 dwt. 4 grs. × 18; 3 oz. 9 dwt. 12 grs. × 41; 7 lbs. 11 oz. 19 grs. × 58.
10. 15 gals. 2 qts. 1 pint × 8; 9 gals. 3 qts. × 18.
11. 35 acres 2 roods 19 poles × 68; 27 acres 19 poles × 26; 7 acres 3 roods 13 poles × 58.
12. 27 leagues 2 miles 6 fur. 2 yds. × 17; 7 miles 2 fur. 28 poles × 59.
13. 5 yds. 3 qrs. 1 nail × 15; 43 yds. 1 qr. 2 nails × 53.

14. Multiply 23 weeks 5 days 7 hours by nine.
Multiply 41 weeks 3 days 27 mins. by forty-two.

15. Multiply 76 qrs. 4 bush. 3 pecks separately by 18, 52, and 21.

16. Multiply £29 18s. 6½d. separately by 14, 29, and 108.

17. Multiply 372 acres 3 roods 29 poles separately by 26, 28, and 54.

18. Find the cost of 117 lbs. at £2 6s. 10½d. per pound.

19. Find the cost of 2300 gallons of spirits at 12s. 8d. per gallon.

20. If an acre of land produce 37 bush. 23 qts., what will 127 acres produce?

21. If I purchase 10 lbs. of cheese at 7½d. per pound, 3 lbs. of tea at 3s. 8d., and 16 lbs. of sugar at 6¾d., what change shall I have left out of two sovereigns?

22. If a draper bought 11 dozen pairs of gloves at 1s. 9½d. per pair, and sold them at 2s. 6d., what did he pay for them, and how much profit did he gain?

23. When a Prussian thaler was worth 2s. 10¾d., what was the value of a bag containing 276 of them?

24. Seventy-five labourers mend a road, at 18s. 6d. per week each; how much will be required to pay them one week's wages?

25. The fare from London to Colchester is 12s. 6d. first class, 9s. second class, and 4s. 3d. third class, and a family of 11 persons travels by first class, with 3 servants, who ride in the second, what is the expense? How much would have been saved if the family had used the second, and the servants the third class carriages?

26. Forty-three persons belonging to a land society receive an allotment of 1 acre 2 roods 23 perches each; what is the total extent of the land?

27. A grocer makes up in one day 57 packages of sugar containing a pound and a half each, 43 containing two pounds, and 75 containing a quarter of a pound. He also sells 25 parcels, each containing a pound and a quarter of tea, 37 containing three quarters of a pound, and 58 containing two ounces each. What is the total quantity of sugar and tea which he sells?

28. A watch gains 6 minutes 13 seconds in a day. How much will it gain in a fortnight?

29. From a nine gallon cask 5½ dozen bottles are filled, each holding three quarters of a pint; how much remains?

30. A tailor bought 12 pieces of cloth, three of them measured 24 yds. 2 qrs. 2 na. each; five others 17 yds. 1 qr. 1 na. each; and the rest 29 yds. 3 qrs. 1 na. each: what was the total length?

SECTION III.—ABBREVIATED METHODS OF MULTIPLICATION.

80. The following methods will occasionally be found of service in shortening the operations of this rule.

CASE I.—WHEN THE MULTIPLIER OR MULTIPLICAND, OR BOTH, HAVE ANY NUMBER OF CIPHERS TO THE RIGHT—

RULE.

Multiply the significant figures only, and add to the product as many ciphers as there are in both factors.

Example.—27000 × 89600.

Here the 7 of the multiplicand means 7000, and the 6 of the multiplier means 600; therefore the factors are 27 × 1000, and 896 × 100. But by (65) these factors may be multiplied in any order; and the product of the 100 and 1000 being 100,000, there remains the product of 27 and 896 only to find: five ciphers added to this product will give the desired answer.

EXERCISE XXI.

1. Multiply 26280 by 45000; 3729000 by 9680000;
2. 14700 by 170; 10740 by 36000; 5209000 by 170.
3. 2874600 by 3970; 52860 by 2900. 4520 by 3300.

81. CASE II.—WHEN THE MULTIPLIER CONTAINS THE FIGURE 1—

RULE.

Add in the figures of the multiplicand at each of those steps of the first partial product under which the same figures would have been placed.

Example.—Multiply 279386 by 18.

I.—Ordinary method. II.—Abbreviated method.

$$\begin{array}{r} 279386 \\ 18 \\ \hline 2235088 \\ 279386 \\ \hline 5028948 \end{array} \qquad\qquad \begin{array}{r} 279386 \\ 18 \\ \hline 5028948 \end{array}$$

As in (I.) each figure of the top line has to be added to that figure of the product which stands one place to its left, it is easier to make this addition without writing down the upper line of figures a second time. See (II.). 8 sixes are 48, set down 8 and carry 4; 8 times 8 are 64, and

4 are 68, and 6 are 74, taking in the last figure of the multiplicand; then 8 × 3 = 24, add 7 *and* 8, the answer is 39; again, 8 times 9 = 72, and 3 *and* 3 = 78; 8 times 7 = 56, and 7 *and* 9 = 72; 8 times 2 = 16, and 7 *and* 3 = 30; 3 *and* 2 = 5. In this way the whole is effected in one line and by a shorter process. The same method applies equally when the 1 is in the hundreds or thousands place.

EXERCISE XXII.

☞ Work the following sums in one line:—

1. 27896 × 17; 30584 × 13; 5196 × 14.
2. 48321 × 15; 70968 × 14; 5096384 × 19; 4078 × 17.
3. 419635 × 16; 210478 × 17; 3121076 × 18; 51943 × 18.
4. 4096 × 19; 123691 × 15; 51427 × 17; 31265 × 18.

Work the following in two lines:—

5. 5081 × 217; 50693 × 189; 71546381 × 231.
6. 718346 × 179; 31456 × 127; 549607 × 518.
7. 21096 × 371; 85462 × 415; 430695 × 163.

Work the following in three lines:—

8. 41863 × 2187; 145963 × 8194; 716384 × 5172.
9. 814065 × 2134; 30796 × 8143; 51968 × 3147.
10. 783260 × 9168; 4718 × 53729; 39682 × 4319.

82. CASE III.—WHEN THE MULTIPLIER NEARLY APPROACHES 100, 1000, 10,000—

RULF.

Add as many ciphers to the multiplicand as there are figures in the multiplier, and subtract from the result the product of the multiplicand and the difference between the multiplier and the 100, 1000, 10,000, as the case may be.

Example.—Multiply 32682 by 999.

$$\begin{array}{rcr}
32682000 &=& 32682 \times 1000\\
\underline{32682} &=& \underline{32682 \times \quad 1}\\
32649318 &=& 32682 \times \ 999
\end{array}$$

Here, because 999 = 1000 − 1, we take the multiplicand a thousand times, and subtract it once, and the required answer is obtained. But to multiply by 1000 (68) we have only to add 3 ciphers.

Multiply 4598 by 997.

$$\begin{array}{rcr}
4598000 &=& 4598 \times 1000\\
\underline{13794} &=& \underline{4598 \times \quad 3}\\
4584206 &=& 4598 \times \ 997
\end{array}$$

Observation.—A similar rule applies to cases in which the multiplier *exceeds* any power of ten, by one. Here the ciphers will be placed as before and the multiplicand *added.* But when the truth of the former method is understood, this will be too simple to require explanation.

EXERCISE XXIII.

☞ Find the following products :—

1. 8096 × 99 ; 54032 × 999 ; 4865 × 9.
2. 7038 × 101 ; 799263 × 8001 ; 171836 × 9999.
3. 54185 × 10001 ; 7218 × 1001 ; 6845 × 103.
4. 21386 × 11 ; 50968 × 99 ; 37254 × 99999.
5. 3876 × 98 ; 4562 × 996 ; 30589 × 95 ; 79264 × 2009.

83. CASE IV.—WHEN ONE PART OF THE MULTIPLIER IS SEEN TO BE A FACTOR OF ANOTHER PART—

RULE.

Multiply first by the less, and afterwards multiply the product by the number of times the greater contains the less. Then add up as usual.

Examples.—Multiply 8427 by 364 ; also 51240 by 742.

I.	II.
8427	51240
364	742
33708 = 8427 × 4	35868000 = 51240 × 700
303372 = 8427 × (90 × 4)	2152080 = 51240 × (6 × 7)
3067428 = 8427 × 364.	38010080 = 51240 × 742.

In (I.) the first line of the product was multiplied by 9 in order to give the second line, because 36 = 9 × 4. In (II.) the first line of the partial product has been multiplied by 6 to give the second line, because 42 = 6 × 7. Caution is needed here to set down the first figure of each answer in the right place. This can only be done by remembering the value of the multiplier in each case, and by applying the rule given in (75).

EXERCISE XXIV.

1. 27963 × 279 ; 85065 × 82 ; 406954 × 963.
2. 7401 × 567 ; 13472 × 39 ; 10968 × 62.
3. 54186 × 486 ; 7096 × 726 ; 2095 × 328.
4. 37264 × 714 ; 51386 × 497 ; 706258 × 369.

SECTION IV.—DESCENDING REDUCTION.

84. REDUCTION is the process of finding how many quantities of one name are contained in a given concrete number of another name.

When it is required to reduce a quantity from a higher to a lower name the process is called *Descending* Reduction. Descending Reduction is performed by Multiplication,

85. *Example I.*—How many farthings are contained in £57? This is simply a sum in Multiplication; for if we know how many farthings are in £1, 57 times that number will give the number of farthings in £57. Or, if we do not know that 960 farthings make £1, but only that 4 farthings are one penny, 12 pence a shilling, and 20 shillings £1, the multiplication of £57 by 20, by 12, and by 4, successively, will produce the same result; thus—

£57
20

1140=shillings in £57, because 20s.=£1.
12

13680=pence in 1140 shillings, because 12 pence=1s.
4

54720=farthings in 13680 pence, because 4 farthings=1d.

Hence 54720 farthings=13680 pence=1140 shillings=£57.

86. *Example II.*—How many pounds are in 41 tons, and How many yards are there in 13 miles?

I. 41 tons.
20

820=cwt. in 4 tons.
4

3280=qrs. in 820 cwt.
28

26240
6560

91840=lbs. in 3280 qrs. or 41 tons.

II. 13 miles.
8

104=furlongs in 13 miles.
40

4160=poles in 104 furlongs.
$5\frac{1}{2}$

20800
2080

22880=yards in 4160 poles.

RULE FOR DESCENDING REDUCTION.*

87. Multiply by as many of the less as make one of the greater; add in to each line all the figures of the original number which refer to the same name, and continue the process, step by step, until the number is reduced to the required denomination.

EXERCISE XXV.

1. How many halfpence in £56 8s. 10½d.? How many farthings?

2. In £105 7s. 8d. how many pence? How many farthings?

3. In £3 8s. 6d. how many shillings? How many tenpences?

4. In £47 10s. how many half-crowns? How many fourpenny pieces?

5. In £125 how many florins? How many pence?

6. How many seconds in 4 lunar months? How many in 2 weeks 3 days? In 7 hours 45 minutes? From 5 P.M. on Tuesday to 7 A.M. on Saturday?

7. How many ounces in 7 cwt.? How many pounds in 3 tons 4 cwt.? How many drams in 3 lbs. 13 oz.? How many grains in 5 oz. 11 dwt.?

8. How many feet in 4 miles? inches in 17 yds. 2 ft.? square feet in 4 acres? square yards in 3 square miles?

9. How many half-pints in 7 gallons? How many pecks in 19 quarters of corn? How many gills in a hogshead of spirits? How many pints in 12 hogsheads of ale?

10. How many more times does the clock tick in March than in February, not being leap-year?

11. A grocer sends to the banker £50 to be changed into sixpences, how many will he have?

---

* It is evident that this Rule, which is usually placed far in advance in books of Arithmetic, has its proper logical position here. First, because no other rule than Multiplication is needed to perform the process; and Secondly, because no sum in Compound Long Division can be worked without Descending Reduction, or without assuming the principles here explained.

SECTION V.—SIMPLE DIVISION.

88. Division is a process—

I. For finding how many times one number contains another.

II. For separating a given quantity into a certain number of equal parts.

III. For finding a number which, if multiplied by another, will produce a third.

IV. For finding the multiplier, when the multiplicand and the product of a Multiplication sum are given.

89. The number to be divided is called the DIVIDEND; that which divides it the DIVISOR;* while the number which shows the times or parts of a time which the Dividend contains the Divisor is called the QUOTIENT.†

Example.—Divide 30 by 5. Here 30 is called the *Dividend*, because it is that which has to be divided. The 5 is called the *Divisor*; and the answer, when found, will be called the *Quotient*.

This question may take several forms; *e.g.*—

I. Find how many times 30 contains 5.

II. Separate thirty into five equal parts.

III. Find what number multiplied by 5 will give 30.

IV. If 30 be the product of two factors, and 5 be one of those factors, what is the other factor?

V. Divide 30 into as many parts as there are units in 5.

VI. Find a number which is as many times less than 30 as unity is less than 5.

VII. How many times can five be subtracted from 30?

90. *Sign of Division* ÷. Thus $12 \div 2 = 6$ (12 *divided by* 2 equals 6). This is more frequently represented by placing the dividend above the divisor and drawing a line between them; *e.g.* $\frac{20}{5} = 4$. In reading such a statement, we commonly say 20 *by* 5 or 20 *upon* 5 equals 4.

---

* From the Latin *divido*, to divide.

† *Quoties* (Latin), how much, or, how many times.

91. Division is a short method of working Subtraction.*

It was shown (60) that multiplication was an abbreviated method of adding the same number to itself a certain number of times, but that Multiplication gives us no help in adding *different* numbers together. Similarly, Division is of service when we wish to subtract the *same* number as often as we can from a greater number; but it will be of no use in any other subtractions. Thus the question just given might be worked in the following manner, and it would then be found that 5 can be subtracted from 30 six times in succession, and leave no remainder; therefore 30 contains 5 exactly 6 times:—

	30	
First	5	
	25	First remainder.
Second	5	
	20	Second remainder.
Third	5	
	15	Third remainder.
Fourth	5	
	10	Fourth remainder.
Fifth	5	
	5	Fifth remainder.
Sixth	5	
	0	

Whoever knows the Multiplication Table well, would, however, be able to tell at once that 6 is the multiplier which makes 5 into 30, and hence that 6 is the answer to the various forms of the question given as an example.

92. From (56) it appears that the multiplier in every Multiplication sum must always be an *abstract* number, because it expresses *how many times* the multiplicand is to be repeated, and cannot refer to a number of articles or things of any kind. Hence in a division sum either the divisor or the quotient must also be an abstract number, although it is not necessary that both should be, for one of these two numbers is always the multiplier, which, if applied to the other, will give the dividend. The form of the question always determines which of the two is abstract. Thus, if we inquire how many times are £6 contained in £48, the quotient will be abstract; but if we ask what the 6th part of £48 is, the divisor is the abstract number. In the former case, £6 is the multiplicand, which, multiplied by 8, will give £48 as answer; and in the latter case £8 is the multiplicand, which, multiplied by 6, the divisor, will produce the dividend. In both cases *one of the two must* be abstract.

---

* It is important that a beginner should be accustomed to look at questions in Division from each of the points of view described in (89); and the teacher will do well to give many exercises on the Multiplication Table, varying the form of the question as shown in the oral exercise, School Arithmetic, p. 23, before the pupil attempts to work out problems in Division in writing.

93. AXIOM IX.—*We divide one number by another when we divide all the parts of the one by the other, and add the answers together.*

Demonstrative Example.—Divide 36 by 3. Now $36=30+6$. If we first divide 30 by 3, and find that the answer is 10, and then divide 6 by 3, and obtain the result, 2, these two answers added together will give the quotient required; *i.e.*—

$$\frac{36}{3}=\frac{30}{3}+\frac{6}{3}=10+2=12.$$

Again: Divide 56 by 4. Suppose we break up 56 into 20, 12, 16, and 8, which are its parts, then—

$$\frac{56}{4}=\frac{20}{4}+\frac{12}{4}+\frac{16}{4}+\frac{8}{4}=5+3+4+2=14.$$

This is not a convenient way; but still the answer might be found in this method, if there were any advantage in breaking up the number into those particular parts.

General Formula.—Whenever $a=b+c+d$,

$$\text{then}\ \frac{a}{x}=\frac{b}{x}+\frac{c}{x}+\frac{d}{x}.$$

94. In Division we always proceed to separate the parts of the dividend in order, beginning with the highest. The right answer *might*, however, be found by beginning with the lower numbers in the dividend, and dividing each of them into the requred number of parts; but in most cases this would prove very inconvenient. In dividing 468 by 2 (I.) it is as easy to begin at one end as another, thus: We may first take the half of 8 and set down 4; then the half of 60, and set down the three tens; and then the half of 400, which gives 200: but in dividing 568 by 2 (II.) after taking the half of 8 and of 60, we have to take the half of 500, which is 250, and then an Addition sum would have to be introduced, and the sum rendered longer than necessary.

$$\begin{array}{l}
\text{I. } 468(2 \\
\hline
\ \ 234
\end{array}
\qquad
\begin{array}{rl}
\text{II. } 568(2 & \\
\hline
4 & = 8\div 2 \\
30 & = 60\div 2 \\
250 & = 500\div 2 \\
\hline
284 & = 568\div 2
\end{array}$$

The ordinary method is as follows:—

Divide 293476 by 7.

$$\text{I.}\quad 7\,)\,\underline{293476}$$

$$\begin{array}{rcr}
40000 & = & 280000 \div 7 \\
1000 & = & 7000 \div 7 \\
900 & = & 6300 \div 7 \\
20 & = & 140 \div 7 \\
5 & = & 35 \div 7 \\
\frac{1}{7} & = & 1 \div 7 \\
\hline
41925\frac{1}{7} & = & 293476 \div 7
\end{array}$$

$$\text{II.}\quad \begin{array}{r} 7\,)\,\underline{293476} \\ 41925\frac{1}{7} \end{array}$$

We here take the figures in order: first the 2, a number of the 6th place, represents 200000, and has to be divided by 7; but this will give no answer in the 6th place; 29 therefore, which is a number of the 5th place, is taken, and is found to contain seven 40000 times, and to leave 10 thousands undivided; these ten thousands, added to the 3 thousands, make 13000, the seventh part of which gives only 1 in the thousands place, and leaves 6000 still to be divided; and 6000 + 400 gives 6400, which, divided by 7, gives 900, and leaves 100. This remainder, added to 70, makes 17 tens, the seventh of which is 2 tens, with a remainder, 3 tens; these, with 6 units, make 36, the seventh of which is 5, leaving 1 remainder. The seventh of this 1 cannot be found, so it is placed at the end of the sum thus $\frac{1}{7}$, and signifies that the division of this 1 has not been effected, but that the answer is 41925, *and the seventh part of* 1.

95. *Observation I.*—It will be seen that, as we ascertained at every step the greatest number which could come into that place of the answer, each figure as it was found might simply have been written in its proper place without the ciphers, and a great deal of trouble saved. Accordingly it is usual to omit all the numbers placed in the example except the last line as in (II.). Beginners, however, should work out a few sums in the former way.

Observation II.—We did not take a seventh part of each number as it stood, but of the *nearest number which contained* an exact number of sevens. Thus the seventh of 280000, not of 290000, was first taken, because it gave an exact 4 in the 5th place, the remaining 10000 being carried forward and accounted for afterwards; so also the seventh of 7000, and not of 13000, was taken, the 6000 being carried on. Again, the seventh of 6400 could not be found in hundreds, so the seventh of 6300 was taken and the remaining 100 carried. In like manner the seventh of 14 tens, and not of 17 tens, was next found; the seventh of 35 units and not of 36 units. The whole dividend has therefore been resolved, for the purpose of dividing it by 7, into the following parts: 280000 + 7000 + 6300 + 140 + 35 + 1. Each of these parts contains an exact number of sevens except the last, and all of them when added together as above make up the whole dividend.

CASE I.—WHEN THE DIVISOR IS LESS THAN THIRTEEN.

RULE.

96. Place the divisor on the left of the dividend (94), find how many times it is contained in the first figure of the dividend, and set down the answer immediately under that figure. If anything remains, carry it and join it to the number on its right; then find how many times the divisor is contained in this number, and set down each figure of the quotient beneath that of the same value in the dividend. Whenever the divisor is not contained at all in the number, write a cipher, to mark that that place is empty, and carry the remainder as before.

Example.—Divide 729346 by 6.

$$6)\ \underline{729346}$$

$121557\frac{4}{6} = 121557$ and the 6th part of 4.

EXERCISE XXVI.

☞ *Note.*—In the case of the first eight sums make a complete analysis as in 94.

1. Divide 72963 by 5; 847256 by 8; 124376 by 11; 12301 by 9; 547229 by 11; 729834 by 6; 271 by 5.

2. $\frac{72981}{7}$; $\frac{86945}{11}$; $\frac{302718}{9}$; $\frac{47216}{8}$; $\frac{82915}{5}$; $\frac{32917}{4}$.

3. $32916 \div 3$; $841729 \div 7$; $729186 \div 12$; $32741 \div 5$; $54729 \div 6$.

4. $\frac{274+8}{5}$; $\frac{56+37-9}{3}$; $\frac{4096+58+4}{7+2}$; $\frac{273+87}{16-5}$.

5. Add the third of 7290 to the sixth of 85572.

6. The sum of £49896 has to be divided among eight persons; how much will each receive?

7. What number multiplied by five will produce 183765?

8. How many hurdles seven feet long will be required to enclose a field which measures 27776 feet in circumference?

97. AXIOM X.—*We divide by the product of two or more numbers when we divide by each of them successively.*

Demonstrative Example.—If we first take the seventh part of a number and then the fourth part of this quotient, we shall evidently have obtained the twenty-eighth part of the original number; for as the seventh is contained 7 times in the whole, the fourth of this seventh is contained 4 times 7 times, or 28 times in the whole. Hence to divide by 4 and by 7 is to divide by 4 times 7, or by 28.

General Formula.—If $a = b \times c$, then $\frac{x}{a} = \frac{x \div b}{c}$.

98. CASE II.—WHEN A DIVISOR IS GREATER THAN TWELVE, AND ITS FACTORS ARE KNOWN—

RULE.

Resolve the divisor into its factors, and divide by them in succession.

Example.—Divide 87423 by 56. Now $56 = 7 \times 8$.

$$
\begin{array}{ll}
7)\underline{87423} & \\
8)\underline{12489} & = 87423 \div 7 \\
\quad 1561\frac{8}{56} & = 87423 \div \overline{8 \times 7} \text{ or } 56.
\end{array}
$$

EXERCISE XXVII.

1. Divide 4362 by 28; 12475 by 45; 962842 by 84.
2. 27961 by 49; 378412 by 120; 23918 by 72.
3. 549 by 21; 62833 by 44; 12719 by 63; 8274 by 25.
4. 39628 by 18; 23718 by 32; 96247 by 99.
5. 82742 by 42; 39728 by 63; 82164 by 56.
6. 72910 by 84; 39721 by 48; 29714 by 108.
7. A has 120 marbles, and B has an eighth as many more. How many have both together?
8. Forty-two persons have £3,906 divided among them; what will the shares of 16 amount to?
9. A tradesman has £1,080, and increases it one-fifteenth at the end of six months, and one-fifteenth at the end of the next half-year. He spends £120 in the year; what will he be worth at the end of the year?
10. What is the difference between the forty-ninth part of 3,087, and the sixty-fourth part of 10,368?
11. If 28 be the divisor, 211 the quotient, and 23 the remainder, what is the dividend?

SECTION VI.—LONG DIVISION.

99. *Example.*—Divide 479632 by 28.

28)	479632	(10000
	280000	7000
28)	199632	100
	196000	20
28)	3632	9
	2800	$\frac{20}{28}$
28)	832	$17129\frac{20}{28}$
	560	
28)	272	
	252	
	20	

Here we take the first figure, 4, but this cannot be divided by 28; we then take forty-seven, and find that this number contains 28 once; but this 47 represents, not units but 47 tens of thousands, and the 1, therefore, in the quotient must be placed or set down to represent 1 ten thousand: 19 tens of thousands are thus left undivided and are carried to the fourth place; with the 9 thousands which stand there they make 199 thousands, which divided by 28 gives 7000, leaving a remainder, 3000, to be carried to the hundreds place. This 30 hundreds + 6 hundreds when divided by 28 gives 100 and leaves a remainder 8: this 800 or 80 tens added to 3 tens makes 83 tens, and this divided by 28 gives 2 tens for the quotient, leaving a remainder 27 tens; 27 tens or 270 if added to 2 give 272, which divided by 28 gives 9 and leaves a remainder 20. This last remainder cannot be divided by 28, so it is set down with the mark of division under it ($\frac{20}{28}$), and this represents the 28th part of 20.

100. *Observation.*—It is necessary again to notice here that the whole dividend has been broken up for convenience into a number of parts, and that each of these has been separately divided by 28.

Now the parts into which the whole has been resolved are—
280000; 196000; 2800; 560; 252; and 20.

And because it has been ascertained that—

280000	contains 28	10000 times
196000	,,	7000 ,,
2800	,,	100 ,,
560	,,	20 ,,
252	,,	9 ,,
20	,,	the 28th part of 20 times.

Therefore 479632 contains 28 17129 and the 28th part of 20 times; or the sum of these partial dividends contains 28, as many times as the sum of the several partial quotients.

RULE FOR LONG DIVISION.

101. Write the divisor to the left of the dividend, and separate them by a line.

Take as many figures from the left of the dividend as there are in the divisor, or if this be a smaller number than the divisor take one more figure of the dividend ; consider this separately as *the first partial dividend.* Find how many times it contains the divisor, place this number on the right as *the first partial quotient.* Multiply the divisor by it, and subtract this product from the first partial dividend, setting down only the remainder.

To this remainder add the next figure of the dividend. Find how many times this contains the divisor, and set down the number thus found as *the second partial quotient.* Multiply as before and subtract the new product from the second partial dividend, setting down the remainder only. To this remainder add the next figure of the dividend, and proceed as before.

Example I.—Divide 53946028 by 253.

I. The process analyzed.

$$\text{Analysis of Process.}\left\{\begin{array}{lcr} & & 253)\ 53946028 \\ 253 \times 200000 & = & \underline{50600000} \\ & & 3346028 \\ 253 \times 10000 & = & \underline{2530000} \\ & & 816028 \\ 253 \times 3000 & = & \underline{759000} \\ & & 57028 \\ 253 \times 200 & = & \underline{50600} \\ & & 6428 \\ 253 \times 20 & = & \underline{5060} \\ & & 1368 \\ 253 \times 5 & = & \underline{1265} \\ & & 103 \end{array}\right. \quad \left.\begin{array}{rcr} (200000 & = & 50600000 \div 253 \\ 10000 & = & 2530000 \div 253 \\ 3000 & = & 759000 \div 253 \\ 200 & = & 50600 \div 253 \\ 20 & = & 5060 \div 253 \\ 5 & = & 1265 \div 253 \\ \underline{\tfrac{103}{253}} & = & \underline{103 \div 253} \\ 213225\tfrac{103}{253} & & 53946028 \div 253 \end{array}\right\}\text{Analysis of Quotient or Answer.}$$

II. The ordinary method.

253) 53946028 ($213225\frac{103}{253}$
506
334
253
816
759
570
506
642
506
1368
1265
103

III. The contracted method.[*]

253) 53946028 ($21322\frac{103}{253}$
334
816
570
642
1368
103

102. *Observation.—The value or place of each figure in the partial quotient is the same as that of the right-hand figure of the partial dividend to which it belongs.* Hence, if after bringing down one figure of the dividend, the partial dividend obtained is found not to contain the divisor, place a cipher in the quotient, bring down another figure of the dividend, and proceed as before.

EXERCISE XXVIII.

☞ A few of these sums should be analyzed in the manner described in Example I.

1. Divide 279684 by 507; 8920562 by 357; 72874 by 36.
2. 509782 by 57; 723486 by 327; 5796843 by 126.
3. 40098765 by 78; 3250796 by 235; 8096274 by 427.
4. 506897 by 525; 829704 by 18.
5. 72843 ÷ 65; 786321 ÷ 49; 52786 ÷ 4098.
6. 306845 ÷ 47; 27839 ÷ 163; 42745 ÷ 87.
7. 309628 ÷ 541; 407864 ÷ 587; 39721 ÷ 675.
8. 5098 ÷ 98; 672543 ÷ 29; 458327 ÷ 123.
9. 40986 ÷ 372; 278437 ÷ 526; 209685 ÷ 324.
10. Find a number equal to the expressions—

$$\frac{29864}{31};\quad \frac{507968}{27};\quad \frac{325+29+5}{47+4};\quad \frac{62870\times 327}{5\times 9}.$$

11. $$\frac{2783\times 6}{17};\quad \frac{1293\times(8+5)}{263};\quad \frac{12086-119}{27\times 3}.$$

---

[*] It is evident that the several remainders written down here are the only figures required afterwards, and that writing the several products in order to subtract them as in II. is for the practised computer unnecessary.

12. $\frac{589+27+163-8}{25+19}$; $\frac{31\times 47}{14+7}$.

13. What is the dividend, if 28 be the divisor, 548 the quotient, and 17 the remainder?

14. If 194 be the quotient, 85 the remainder, and 784621 the dividend, what is the divisor?

15. What number multiplied by 173 will make 9754432?

SECTION VII.—COMPOUND DIVISION.

103. It has been seen that in Simple Division we begin with numbers of the highest value, divide them, and after finding the largest possible quotient in a number of the same value, carry on the remainder, reduce it from thousands into hundreds, or from hundreds to tens, and then find the greatest quotient in this lower name. So if instead of thousands, hundreds, tens, and units, the dividend consist of pounds, shillings, pence, and farthings, it will be necessary to find the quotient in pounds, then carry on the remainder, and reduce it to shillings, afterwards to divide the shillings and carry on the remainder to the pence, and so on. (84). In the following examples it will be seen that the two processes are identical in principle :—

I. Divide 23733 by 6 ;

I. 6) 23733 (3 thousands
18
5 thousands
10
6) 57 (9 hundreds
54
3 hundreds
10
6) 33 (5 tens
30
3 tens
10
6) 33 (5 units
30
3 remainder.

Answer — 3 thousands, 9 hundreds, 5 tens, 5 units, and the sixth part of 3 ; or $3955\frac{3}{6}$.

II. Divide £23 7s. $3\frac{3}{4}$d. by 6.

£ s. d.
II. 6) 23 7 $3\frac{3}{4}$ (3 pounds.
18
5 pounds
20
6) 107 (17 shillings
102
5 shillings
12
6) 63 (10 pence
60
3 pence
4
6) 15 (2 farthings
12
3 farthings remainder.

Answer — 3 pounds, 17 shillings, 10 pence, 2 farthings, and the sixth part of 3 farthings ; or £3 17s. 10 d. $2\frac{3}{6}$ farthings.

104. Although we do not in practice set down 10 as a multiplier, as shown in the former of these examples, yet we mentally make the same reduction as if we did so. Thus, after finding the sixth of 23 thousands to be 3 thousands and leave 5 thousands remainder, we bring down the other figure to its side, and treat the new number, 57, by itself. But this 57 is of a lower name than thousands; the 5 thousands have really been converted into 50 hundreds, and added to the other hundreds of the dividend. The next remainder, 3 hundreds, is reduced in like manner to 30 tens, and added to the 3 tens which already form part of the dividend.

This method of reducing the remainder of each denomination or place into equivalent numbers in the lower name or place, is employed throughout the whole of Simple Division, and is equally applicable to Compound.

In the second of the examples the number of pounds remaining after the first quotient is obtained is reduced to shillings and added to the other shillings of the dividend. A new division is then made of shillings only, the remainder reduced to pence, and treated in the same way.

Example.—What is the fifteenth part of 23 tons 17 cwt. 2 qrs. 5 lbs. 3 oz.?

tons. cwt. qrs. lbs. oz.
15) 23 17 2 5 3 (1 ton 11 cwt. 3 qrs. 9 lbs. $10\frac{13}{15}$ oz.
15
8
20
15) 177 (11 cwt. = 165 cwt. ÷ 15
165
12
4
15) 50 (3 qrs. = 45 qrs. ÷ 15
45
5
28
15) 145 (9 lbs. = 135 lbs. ÷ 15
135
10
16
15) 163 (10 oz. = 150 oz. ÷ 15
150
15) 13 (the fifteenth part of 13 oz.

Ans.—1 ton 11 cwt. 3 qrs. 9 lbs. 10 oz. and the fifteenth part of 13 ounces.

RULE FOR COMPOUND DIVISION.

105. Divide the numbers of the highest name as in Simple Division. By the Rule of Reduction (87) bring the remainder and the next figure into equivalent numbers of the next lower name. Make a new Division sum with this number, and set down the answer by itself. Reduce the remainder into numbers one step lower, and divide this new number separately. Proceed in the same way with the remainders until the lowest is reached.

106. *Observation.*—No new truth of Arithmetic is wanted or illustrated here. Axiom IX. tells us that it is allowable to break up the dividend in any manner which may be most convenient, and to divide it part by part. And it has been seen (94) that it is more convenient to begin with the greater numbers.

Exercise XXIX.

1. Divide £723 11s. 8d. by four; £52 10s. 4½d. by six;
£293 10s. 8¼d. by 3.
2. Find the fifth of £297 16s. 3d.; the sixth of £274 13s. 8d.; and
the seventh of £1000.
3. Find the 7th, the 8th, and the 9th of £6274 15s. 8½d.
4. £297 14s. 3¼d. ÷ 28; £3654 10s. 11d. ÷ 352.
5. £1270 3s. 7d. ÷ 514; £1560 3s. ÷ 17; £2740 3s. 1¼d. ÷ 510.
6. £12783 13s. 9d. ÷ 60; £3185 3s. 6½d. ÷ 137;
£2186 15s. 3d. ÷ 49.
7. £29841 13s. 7d. ÷ 631; £62108 17s. 9¼d. ÷ 65;
£270 10s. ÷ 135.
8. 73 cwt. 1 qr. 26 lbs. ÷ 15; 21 tons 5 cwt. 3 qrs. ÷ 29;
7 cwt. 25 lbs. ÷ 5.
9. 6 lbs. 3 oz. 4 dwt. ÷ 8; 15 lbs. 11 oz. 7 grs. ÷ 14;
29 lbs. 3 dwt. 14 grs. ÷ 125.
10. 17 miles 2 fur. 17 poles ÷ 16; 192 miles 6 fur. ÷ 23;
7 fur. 18 poles 3 yds. ÷ 14.
11. Find a 4th, a 6th, an 11th, and a 13th of a field of 17 acres.
12. 372 qrs. 2 bush. 1 peck ÷ 8; 17 qrs. 7 bush. 3 pecks ÷ 27;
15 qrs. 6 gals. ÷ 9.
13. Divide £4965 10s. 3¾d. equally among 63 persons.

107. In Compound Division the divisor is generally abstract, and the quotient consists of concrete numbers of the same kind as the dividend. It is evident (89) that the word quotient does not in strictness apply to such an answer as this.

108. Sometimes, however, a concrete number has to be divided by another similar concrete number.

Example.—How many times is 2¼d. contained in £43 18s. 7d.? Here the answer will be an abstract number.

Now because, by Rule of Descending Reduction (87), 2¼d. contains 9 farthings, and £43 8s. 7d. contains 42172 farthings, the question takes this form :—

How many times are 9 farthings contained in 42172 farthings?
Or, how many times are 9 contained in 42172?

By Simple Division $\frac{42172}{9} = 4685\frac{7}{9}$ = the answer.

109. The two numbers must always refer to magnitndes of the same name. It would be absurd to ask how many times 3 miles were contained in 17 days.

It should also be noticed that although we may divide one concrete number by another, we cannot multiply one concrete number by another. The question, for example, Multiply £17 3s. by £17 3s. is evidently unmeaning. For of two factors the multiplier must *always* be an abstract number (56), but of divisor and quotient, either may be concrete, provided the other is abstract.

RULE FOR DIVIDING ONE CONCRETE NUMBER BY ANOTHER OF THE SAME KIND.

110. Reduce both to the same name* (the highest to which both can be reduced) and divide the greater by the less.

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* Without this precaution, the divisor would be needlessly large, and the difficulty of working increased.

EXERCISE XXX.

1. £17 3s. 9d. ÷ 3¾d. £965 14s. 3½d. ÷ 2s. 6¼d. ;
£87 4s. 1½d. ÷ 7½d.

2. £392 10s. 3½d. ÷ 1s. 7¼d. ; £62 10s. ÷ 1s. 3d. ;
£274 18s. 5d. ÷ 4¾d.

3. £1027 5s. 6d. ÷ £2 3s. 8d. ; £453 12s. 8¼d. ÷ 2s. 9¼d. ;
£62 3s. 5d. ÷ 1s. 8½d.

4. 23 lbs. 15 oz. ÷ 2 oz. 8 drs. ; 17 cwt. 3 qrs. ÷ 17 lbs. 3 oz. ;
196 cwt. ÷ 2 lbs. 11 oz.

5. 17 miles 3 fur. 18 poles ÷ 29 yds. ; 23 miles 6 fur. ÷ 6 yds. 2 ft. ;
412 miles 6 fur. 18 poles ÷ 3 miles 2 fur.

6. 5 years 186 days ÷ 17 hours 3 mins. ;
27 weeks 4 days ÷ 3 days 6 hours.

7. 17 acres 3 roods 29 poles ÷ 2 roods 7 poles ;
5166 acres 2 roods 14 poles ÷ 5 acres 3 roods ;
29 acres 1 rood 13 poles ÷ 39 poles.

8. 17 bush. 2 pecks ÷ 5 pints ; 2974 qrs. ÷ 1 bush. 1 peck.
728 qrs. 3 bush. 1 peck ÷ 2 qrs. 3 pecks.

9. How many coins worth 2¾d. each are equal in value to £25?

10. How many fourpenny pieces and sixpences are there in £28 17s. 6d.?

11. How many guineas are there in £7934?

12. Reduce £1738 to half-guineas, and the same number of half-guineas to pounds.

13. How many packages of 11 oz. are made from 2 qrs. 18 lbs.?

14. How many allotments of 18 poles each can be made from 19 acres 3 roods?

15. How often will a hoop 2 ft. 3 in. in circumference revolve in rolling 3 miles?

16. How many periods of 7 minutes 17 seconds each occur in 11 days and nights?

17. A man walks 7½ furlongs in 15 minutes; how long will he be walking 19¾ miles?

18. How many florins, worth 1s. 8½d. each, are equivalent in value to 24 five-pound notes?

19. A schoolroom is 27 feet square. How many pieces of paper, each 17 inches by 14, would cover the floor?

20. In 100 years how many more days has Jupiter than the earth, the length of Jupiter's day being 9 hours 56 minutes; a year being reckoned as 365¼ days?

METHODS OF PROVING MULTIPLICATION AND DIVISION.

111. The connection already explained (88) between Multiplication and Division shows that either rule may be used to prove the other. For because the answer to a Multiplication sum is produced by multiplying a given number by a second, it follows that to divide this answer by the second is to reproduce the original multiplicand.

TO PROVE MULTIPLICATION—RULE.

112. Divide the answer by the multiplier, and if it gives the multiplicand the sum is right.

EXERCISE XXXI.

☞ Prove the first twelve sums in Exercise XVI. by this rule.

A similar method applies to Division, because (88) the quotient sought is always such a number that if it be used as a multiplier and applied to the divisor it will produce the dividend. The remainder, however, if any, being *that portion of the dividend which continues undivided*, must be added to the product of divisor and quotient.

TO PROVE DIVISION—RULE I.

113. Multiply the quotient by the divisor, add the remainder to the result, and if the answer equals the dividend the sum is right.

EXERCISE XXXII.

☞ Prove the sums in Exercise XXVI. (1 and 2) by this method.

TO PROVE DIVISION—RULE II.

114. Select from the sum, after working it, all the partial dividends which have been taken, including the remainder. Add these together, and if the answer equals the dividend the sum is right.

Observation.—This has been already explained (100) as a process of analysis, it is also a method of proof.

EXERCISE XXXIII.

☞ Prove sums in Exercise XXVI. (3 and 4) by this method.

115. *Casting out nines.* The reason of this rule will be found in the Section on Special Properties of Numbers (175).

I. To prove Multiplication.

$$\begin{array}{rl} 7963 & 7 \\ \underline{58} & \underline{4} \\ 63704 & 28 = 9 \times 3 + 1 \\ \underline{39815} & \\ 461854 & 1 \end{array}$$

Throw out the nines of the multiplicand. (In Example, 7+9+6+3 = 25 = 2×9+7, *set down* 7.) Do the same with the multiplier. (In Example, 5+8=13=9+4, *set down* 4.) Do the same also with the answer to the sum (4+6+1+8+5+4 = 28 = 3×9+1, *set down* 1). Now, if the product of the two former remainders (7 and 4) divided by 9 leaves this last remainder (1), the sum is likely to be right; but if not it is certainly wrong.

EXERCISE XXXIV.

☞ Prove the first 12 sums in Exercise XXVII. by this method.

II. To prove Division.

$$\begin{array}{rrl} 2 & & \;\;\;\;\;\;\;6 \\ 29) & 683475 & (23568 \\ & 103 & \\ & 164 & \\ & 197 & \\ & 235 & \\ & 3 & \end{array}$$

Throw out the nines from the divisor and quotient. (In Example 2 + 9 = 11, *set down* 2 over divisor; and 2 + 3 + 5 + 6 + 8 = 24 = 2 × 9 + 6, *set down* 6 over quotient.) Then subtract the remainder from the dividend and add up the figures which are left. (In Example, 6 + 8 + 3 + 4 + 7 + 5 − 3 = 30.) Divide this by 9, and if the remainder is the same as that obtained by dividing the product of the other two numbers by 9 the answer is likely to be right; but if not it is certainly wrong. (In Example, 2 × 6 = 9 + 3, and 30 = 3 × 9 + 3.)

EXERCISE XXXV.

☞ Prove sums in Exercise XXVIII. (1 and 2) by this method.

SECTION VIII.—ASCENDING REDUCTION.

116. Reducing a concrete number to an equivalent number of a higher name is called ASCENDING REDUCTION.

117. Ascending Reduction is to be worked by Division.

Example.—How many days are in 1000000 seconds? Here, if we know the number of seconds in a day, and divide 1000000 by it, we may at once ascertain the number of days. But the same result may be reached by successive steps.

$$\begin{array}{r l} 60)\underline{1000000} & \\ 60)\underline{16666} & 40 \\ 24)\underline{277} & 46 \\ 11 & 13 \end{array}$$

Because there are 60 seconds in one minute, therefore there are one sixtieth part of 1000000 minutes in 1000000 seconds. Hence 16666 minutes and 40 seconds = 1000000 seconds. But because there are 60 minutes in one hour, there are one-sixtieth part of 16666 hours in 16666 minutes. Hence 277 hours 46 minutes = 16666 minutes. But again because there are 24 hours in a day, there are one-twenty-fourth of 277 days in 277 hours. Hence 277 hours = 11 days 13 hours.
Therefore we have found successively that—

1000000 seconds = 16666 minutes 40 seconds.
= 277 hours 46 minutes 40 seconds.
= 11 days 13 hours 46 minutes 40 seconds.

RULE.

118. Divide the number by as many of the less quantity as make one of the next greater. Leave the remainder, and divide the first quotient by as many of its denomination as make one of the next above it. Leave the remainder and continue to divide each new quotient until the highest is reached.

Observation.—The remainder is that which remains of a particular dividend, and is always therefore of the same kind and name.

Example I.—Reduce 2534 farthings into pounds.

4) 2534 farthings
12) 633 2 = 633 pence + 2 farthings.
20) 52 9 = 52 shillings 9 pence.
2 12 = 2 pounds 12 shillings
∴ 2534 farthings = £2 12s. 9½d.

II.—Reduce 4398 grains into pounds troy.

24) 4398 grains.
20) 183 6 = 183 dwt. 6 grs.
9 3 = 9 oz. 3 dwt.
∴ 4398 grains = 9 oz. 3 dwt. 6 grs.

EXERCISE XXXVI.

1. Reduce 1233 farthings to pounds, and 171923 halfpence to pounds.

2. How many pounds in 27963 farthings, and in the same number of halfpence and pence?

3. Reduce 23856 seconds to hours; 158960 minutes to days; and 10005000 hours to years.

4. How many crowns in 583721 farthings?

5. Reduce 270628 feet to miles, and 863 inches to yards.

6. What is the number of tons in 447826 ounces?

7. By how much does the number of gallons in 10000 pints exceed the number of hogsheads in 10000 gallons?

8. Reduce 7098 grains to Troy pounds, and the same number to Avoirdupois pounds.

9. A side of a square measures 647 feet, what is its area?

10. What is the difference between the number of acres in a parallelogram, one of whose sides is 738 yards and the other 514 yards, and those in a square whose side measures 1720 feet?

11. How many square feet are required to glaze 5 windows, each containing 14 panes of glass, the panes measuring 17 in. by 15 in. each?

12. If a piece of ribbon measures 21 yards 2 nails, how many bonnets can be trimmed with 17 such pieces, suppose each bonnet require 2½ yards?

13. How long would it take to count a million of millions of money if £100 were to be counted every minute without intermission, the year to consist of 365 days, 5 hours, 49 minutes?

14. In 53777630 drams how many tons?

15. In 4712 inches of linen, how many yards?

16. In 648385 grains of gold, how many pounds?

** SECTION IX.—MULTIPLICATION AND DIVISION.

119. MULTIPLICATION.—The relations between multiplier, multiplicand, and product, in the former of these rules, and between the corresponding terms, quotient, divisor, and dividend, in the latter rule, are so important, that it is desirable to examine them somewhat more closely.[*]

120. AXIOM XI.—*We increase or diminish the product when we increase or diminish the multiplier.*

Demonstrative Example.—If we have a number, say 26, to be multiplied by 6, and obtain a certain answer, it is evident that multiplying by 3 times 6 would give three times that answer.

Again, if instead of multiplying by 6 we multiply by the third part of 6, we obtain only one-third of the product.

General Formula.—If $a \times b = c$, then $a \times mb = mc$,

and $a \times \frac{b}{m} = \frac{c}{m}$.

121. AXIOM XII.—*We increase or diminish the product when we increase or diminish the multiplicand.*

Demonstrative Example.—Suppose on multiplying one number by a second we obtain a certain answer, multiplying ten times that number by the second would give ten times that answer; and similarly, multiplying a tenth part of that number by the second would give only a tenth part of that answer.

General Formula.—If $a \times b = x$, then $na \times b = nx$,

and $\frac{a}{n} \times b = \frac{x}{n}$.

The result of the two last axioms may be generally expressed thus :—

122. *If we increase or diminish either of the factors a certain number of times we make the same increase or diminution in the product.* This is sometimes expressed by saying that the product *varies as* the multiplicand or multiplier.

123. *If one factor is increased as many times as another is diminished the product remains unaltered.*

Demonstrative Example.—If instead of multiplying 8 by 6 we multiply twice 8 by the half of 6, or if we multiply three times 8 by the third of 6, we obtain exactly the same answer. For the increase of the one factor, which tends to make the product a certain number of times greater, destroys the effect of the decrease of the other, which tends to make the product the same number of times less.*

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[*] For the right understanding of the more advanced portion of the science, the axioms here stated are indispensable, and they will frequently be alluded to in future demonstrations.

Thus 7 times 12 = $(3 \times 7) \times \frac{12}{3}$ or $(4 \times 7) \times \frac{12.}{4}$†

General Formula.—$(a \times b) = na \times \frac{b}{n}$.

EXERCISE XXXVII.

☞ Place a factor in each of the vacant spaces, so that the products shall be equal.

$5 \times 9 = \frac{9}{3} \times (\quad)$; $16 \times 12 = (\quad) \times \frac{12}{4}$; $49 \times 3 = \frac{49}{7} \times (\quad)$.

$18 \times 6 = \overline{2 \times 6} \times (\quad)$; $36 \times 8 = \frac{36}{4} \times (\quad)$; $350 \times 17 = \frac{350}{10} \times (\quad)$.

$a \times b = 3a \times (\quad)$; $x \times y = \frac{x}{a}(\quad)$; $p \times q = np \times (\quad)$.

124. *Corollary I.—If there be three numbers so related that the first is as many times greater or less than the second, as the second is greater or less than the third, the product of the first and third will equal the second multiplied by itself.*

This is only a particular application of the truth expressed in (123).

Thus: 7, 14, 28. Here 14 = 2 × 7, and 28 = 2 × 14. But (123) 14 times 14 = the half of 14 × twice 14, or 7 × 28.

Again, in the numbers 3, 12, 48. Because 12 is as many times more than 3 as it is less than 48, therefore 12 × 12 = 3 × 48; and generally,

$$\left.\begin{array}{l}\text{If } a \text{ is as many times greater than } b, \text{ as } b \text{ is greater than } c\\ \qquad\qquad\qquad\qquad\text{or,}\\ \text{If } a \text{ is as many times less than } b, \text{ as } b \text{ is less than } c \ \ldots\ldots\ldots\end{array}\right\} \text{then } ac = bb.$$

---

* The principle here explained suggests to us another method of proving Multiplication, although it is too cumbersome to be often used. Suppose I have 586 to multiply by 36, and after doing it wish to verify the answer; twice 586 multiplied by the half of 36 would produce the same result; so would 6 times 586 multiplied by the *sixth* part of 36.

† Although the form employed here ($\frac{12}{4}$) appears fractional, it is necessary to remind teachers that we are here only concerned with it as far as it implies *division*, It will of course be read "the fourth part of 12." Throughout the whole of this chapter similar expressions should be read in the same way. The teacher will see that no one definition or principle of "Fractions" is anticipated by this arrangement although when a pupil arrives at Fractions it will be a great advantage to him to be already familiar with such forms under the name of divisor and dividend.

EXERCISE XXXVIII.

☞ Place a number in each vacant space, so that the product of the first and third shall equal the second multiplied by itself.

1. 3, 12 ( ); 2, 14 ( ); 18, 6 ( ).
2. 36, 18 ( ); 27, 9 ( ); 7, 21 ( ).
3. 25, 5 ( ); 72, 144 ( ); 18, 54 ( ).
4. a, $3a$ ( ); x, nx ( ); np, p ( ).

125. If the product of two numbers equals the product of two others, and any three of these are known, the fourth may be found by dividing the product of the given pair by the remaining number.

Because 16 × 3 = 4 × 12, the 16th part of 4 × 12 will equal 3; the 3rd part of 4 × 12 will give 16; the 4th part of 16 × 3 will give 12; and the 12th part of 16 × 3 will give 4. If any one of these had been unknown it might have been found thus:—

$$12=\frac{16\times 3}{4};\quad 4=\frac{16\times 3}{12};\quad 16=\frac{4\times 12}{3};\quad 3=\frac{4\times 12}{16}.$$

General Formula.—Whenever $ab=cd$,

$$\text{then } a=\frac{cd}{b};\quad b=\frac{cd}{a};\quad c=\frac{ab}{d};\quad d=\frac{ab}{c}.$$

EXERCISE XXXIX.

☞ Place a number in each vacant space, so that the product shall be equal in each of the following cases.

1. 16×8=2× ( ); 27× ( )=4×9.
2. 81×4=12× ( ); 542×36=391× ( ).
3. 4726×9=27× ( ); 5089×63=716× ( ).
4. 16×7192=314× ( ); 154×379=812× ( ).
5. 7246×372=128× ( ); 100×5=75× ( ).
6. 36×81=243× ( ); $a\times b=c\times$ ( ).

126. *Corollary II.—If there be four numbers so related that the first is as many times more or less than the second, as the third is more or less than the fourth, the product of the second and third will equal that of the first and fourth.*

For in this case the second and third would give a certain product; but of the other two one is as many times greater than the second as the other is less than the third, or one is as many times less than the second as the other is greater than the third, hence (123) the two products must be equal.

Example.—5, 20, 3, 12. Here the second equals 4 times the first, and the fourth 4 times the third. Because the second is as many times greater than the first as the third is less than the fourth, $20 \times 3 = 5 \times 12$; *i. e.*, 20 times 3 = the fourth of 20×4 times 3.

EXERCISE XL.

☞ Illustrate the foregoing propositions by placing a suitable number in each of the vacant spaces.

1. 5 15 7 ( )
2. 5 20 16 ( )
3. 10 ( ) 4 12
4. 90 ( ) 18 2
5. 17 ( ) 9 18
6. $3a$ a b ( )
7. 25 5 75 ( )
8. $\frac{x}{4}$ x y ( )
9. pq p ( ) m
10. ( ) b a $n \times a$

It may be easily inferred from the last paragraph that—

127. *If there be a series of numbers, arranged in order by regular and equal multiplication or division, they may be formed into a number of pairs of factors, each pair giving the same product.*

Thus, in the following series, in which each number is 3 times that on its left,

$$\overbrace{12 \quad \overbrace{36 \quad \overbrace{108 \quad 324} \quad 972} \quad 2916},$$

108 multiplied by 324 would give a certain product.

But 36 is the third of 108, and 972 equals three times 324.

Therefore $108 \times 324 = 36 \times 972$.

So also 12 is as many times less than 108 as 2916 is more than 324. Therefore $12 \times 2916 = 108 \times 324$.

In the same manner any two numbers may be taken in such a series, and their product will be found equal to that of any other two at equal distances from the first two.

128. From (125) we infer that the product of any two numbers at equal distances from any *one* of such a series will equal that one multiplied by itself. For by the hypothesis every figure on one side of it will be as many times less than the number as that in the corresponding place on the other side will be greater; hence the product of the two numbers so chosen will be equal to the second power, or square, as it is called, of the central number.

EXERCISE XLI.

☞ Find how many equal products can be made out of each of the following series:—

1. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024.
2. 3, 6, 12, 24, 48, 96, 192.
3. 1, 5, 25, 125, 625, 3125, 15625.
4. 32, 16, 8, 4, 2, 1, ½.
5. 81, 27, 9, 3, 1.

129. DIVISION.—It is necessary to remember that in every Division sum the dividend is the product, of which the divisor and the quotient are the two factors, and that because—

In Multiplication, multiplier × multiplicand = product;
In Division quotient × divisor = dividend;

Therefore every truth which could be asserted respecting Multiplication can be asserted in another form which will be applicable to Division.

130. AXIOM XIII.—*If we increase or diminish the dividend a certain number of times, we make the same alteration in the quotient.*

Demonstrative Example.—If 28 divided by 4 will give a certain answer, 3 times 28 divided by 4 will give 3 times that answer, and the half of 28 divided by 4 will give the half of that answer; or

General Formula.—If $\frac{a}{b} = x$, $\frac{na}{b} = nx$, and $\frac{a \div n}{b} = \frac{x}{n}$.

Observation.—This is the converse of (122). It was there stated that increasing one of the factors increased the product; here it is asserted that increasing the product makes necessary the increase of one of the factors. For the quotient being one of the factors, and the other (the divisor) remaining the same, whatever increases the product (the dividend) makes a corresponding increase necessary in the other factor. Similarly, it may be seen that if the dividend has been diminished while the divisor remains the same, the other factor (the quotient) must also be diminished as much as the dividend has been. This is sometimes expressed by saying that the quotient *varies as* the dividend.

131. AXIOM XIV.—*If the divisor be increased a certain number of times, the quotient is diminished in the same degree; but if the divisor be diminished, the quotient is increased.*

Demonstrative Example.—If 270 divided by 15 gives a certain answer, 270 divided by twice 15 would give one-half of that answer; and 270 divided by a fifth of 15 would give 5 times that answer.

General Formula.—If $\frac{a}{b} = x$, then $\frac{a}{nb} = \frac{x}{n}$, and $\frac{a}{b \div n} = nx$.

This assertion is the converse of that in (123); for because in a Division sum the dividend is the product, of which the divisor and quotient are the factors, if we retain the dividend, and diminish the divisor, we must have a greater quotient; while the same dividend, and a greater divisor, must have a smaller quotient.

The truth contained in the two last axioms may be concisely expressed thus :—

132. The answer to a Division sum is made greater by increasing the dividend or by diminishing the divisor; but it is made less by diminishing the dividend or by increasing the divisor.

133. AXIOM XV.—*If the dividend and divisor be either both increased or both diminished the same number of times, the quotient remains unaltered.*

Demonstrative Example.—If 42 divided by 6 gives 7, twice 42 divided by twice 6 will give the same answer; ten times 42 divided by ten times 6; the half of 42 divided by the half of 6; or the third of 42 divided by the third of 6 will give the same answer.

General Formula.—If $\frac{a}{b} = x$, $\frac{an}{bn} = x$, and $\frac{a \div n}{b \div n} = x$.

Observation.—In (122) it was stated that if one factor remained the same while the other was increased or diminished, the product was increased or diminished in the same degree. It is equally evident that if the product and one factor are both increased or both diminished in the same degree, the other factor must remain the same.

EXERCISE XLII.

☞ Find three other pairs of divisors and dividends which would give the same quotient as each of the following :—

1. $\frac{290}{10}$ 47 ÷ 3; 29 ÷ 6.
2. 58 ÷ 4; 572 ÷ 12; 435 ÷ 8.
3. 48 ÷ 6; 59 ÷ 12; 523 ÷ 6.
4. 15 ÷ 3; 498 ÷ 20; 135 ÷ 16.

The foregoing principles are seen to be of practical utility in the following cases:—

134. CASE I.—WHEN BOTH DIVIDEND AND DIVISOR TERMINATE IN ONE OR MORE CIPHERS—

RULE.

Cut off the same number of ciphers from both, and work the division with the numbers thus diminished.

For to cut off a cipher is to divide by 10; to cut off two, three, or four ciphers is to divide by 100, 1000, or 10,000. But if this be done to both, the sum will be more readily worked, and by (133) the quotient will be the same as if the numbers had not been altered.

EXERCISE XLIII.

☞ Work the following sums by the contracted method:—

1. $58600 \div 700$; $273000 \div 18000$; $396180 \div 500$.
2. $3500 \div 400$; $5169000 \div 800$; $37290 \div 90$.
3. $7200 \div 100$; $51980 \div 60$; $573000 \div 27000$.

135. CASE II.—WHEN THE DIVISOR IS COMPOSED OF A WHOLE NUMBER AND A PART—

RULE.

Multiply both divisor and dividend by such a number as will convert the divisor into a whole number, and then perform the division.

Example.—Divide 573 by $3\frac{3}{4}$.

Here we choose 4 because it is the multiplier which, applied to $\frac{3}{4}$, will convert it into a whole number.

$$(573 \times 4) \div (3\tfrac{3}{4} \times 4) = 573 \div 3\tfrac{3}{4}.$$

But $573 \times 4 = 2292$, and $3\frac{3}{4} \times 4 = 15$;

$$\therefore \frac{573}{3\frac{3}{4}} = \frac{2292}{15} = 152\tfrac{12}{15}.$$

EXERCISE XLIV.

☞ Work the following sums:—

1. $463 \div 2\frac{1}{2}$; $827 \div 5\frac{1}{2}$; $1006 \div 8\frac{1}{2}$.
2. $587 \div 4\frac{1}{3}$; $327 \div 2\frac{3}{8}$; $789 \div 5\frac{3}{4}$.
3. $6908 \div 2\frac{1}{4}$; $5172 \div 4\frac{1}{2}$; $3976 \div 6\frac{1}{4}$.
4. $12506 \div 5\frac{1}{3}$; $1089 \div 5\frac{1}{3}$; $1627 \div 7\frac{3}{8}$.
5. $1396 \div 4\frac{2}{3}$; $4096 \div 3\frac{2}{3}$; $8162 \div 4\frac{1}{5}$.
6. $52746 \div 6\frac{3}{8}$; $3825 \div 9\frac{1}{4}$; $4568 \div 7\frac{2}{3}$.

136. CASE III.—WHENEVER THE PRODUCT OF ANY GROUP OF NUMBERS HAS TO BE DIVIDED BY THE PRODUCT OF ANOTHER GROUP, AND THE TWO GROUPS CONTAIN COMMON FACTORS—

RULE.

Strike out the factors common to dividend and divisor, and perform the division with the remaining products only.

This operation is called *Cancelling.*

Example.—Divide $75 \times 6 \times 8 \times 4$ by $5 \times 9 \times 8 \times 6$. Here the 8 and 6 occur in both, and 75 may be resolved into 15×5. Hence 8 and 6 and 5 may be rejected, because we thus divide both dividend and divisor by the same numbers, and (133) this does not alter their product. Hence—

$$\frac{75 \times 6 \times 8 \times 4}{5 \times 9 \times 8 \times 6} = \frac{75 \times 4}{5 \times 9} = \frac{(15 \times 5) \times 4}{5 \times 9} = \frac{15 \times 4}{9} = 6\frac{6}{9}$$

EXERCISE XLV.

Simplify the following by cancelling, and find the answers :—

1. $\frac{5 \times 4 \times 3}{9 \times 6 \times 5}$; $\frac{28 \times 12}{7 \times 12}$; $\frac{27 \times 4 \times 9}{3 \times 8 \times 9}$.
2. $(214 \times 6) \div (27 \times 6)$; $(23 \times 5 \times 108) \div (4 \times 5 \times 12)$.
3. $\frac{7 \times 8 \times 9}{3 \times 8 \times 6}$; $\frac{20 \times 12 \times 48}{24 \times 8 \times 10}$; $\frac{b \times c \times d}{a \times x \times b \times c}$.
4. Divide the product of 18, 16, 15, and 4, by that of 20, 25, 30, and 12.
5. Divide the product of 3, 17, 18, 24, and 100, by that of 34, 9, 36, and 40.

Questions on Multiplication and Division.

Define Multiplication, multiplicand, multiplier, product, and factor. What is a square? How can Multiplication be said to be abridged Addition? What is a distributive operation? Express the product of seven and nine in four different ways, showing of what parts the whole is made up.

State the first axiom assumed in Multiplication. Give an example. State the second, and give an example. What rule may be deduced from the second axiom? If in multiplying by 7000 we multiply by 7 and add three ciphers, what principles are illustrated by the process?

What is Long Multiplication? State the rule, and the truth on which the rule depends. What method is always employed in Compound Multiplication? Why is no other method allowable? State how a sum may be shortened when such a number as 218, 98, 101, or 369 is the multiplier.

Define Reduction. How much of Reduction may be performed by Multiplication? Why? Give an example and state the rule.

What is meant by Division, dividend, quotient, divisor? In how many different ways can you state the problem $a \div b$? What sort of Subtraction sums may be abridged by the Rule of Division? What is the first axiom assumed in Division?

State the rule for Simple Division. Why is it better to commence at the left hand of the sum? How do you divide by a large number when its factors are known? Repeat the truth assumed in this rule.

Give the Rule for Long Division. What is meant by partial dividends and quotients? If all the partial dividends be added up, what should be their sum? How does the Rule for Descending Reduction apply to Compound Division? When is the answer to a Division sum really a quotient? When is it not so?

State the methods of proving Multiplication and Division. Show how Ascending Reduction belongs to Division, and why? How does the increase of the multiplier or multiplicand affect the product? State the axioms on this subject.

What change may take place in the factors which will have no effect on the product? Give an example. What inferences may be drawn from the truth you illustrate? Give examples.—I. Of three numbers, the product of the first and third equalling the second multiplied by itself; and II. Of four numbers, having the product of the first and fourth equal to that of the second and third. State the reason in each case.

Construct a series of numbers such that the first and last shall give the same product as any pair equally distant from them, and give the reason. In Division what is the effect of increasing or diminishing the dividend?—the divisor?

What changes can be effected in the divisor and dividend which will not affect the quotient? State the truth in words, and give an example. What practical uses arise out of this principle? Explain cancelling, and give an example.

State in words the truth expressed by each of the following formulæ:—

1. If $x = y + z$, then $mx = my + mz$.
2. $xyz = zxy$.
3. If $z = xy$, then $mz = (m \times x) \times y$.
4. If $m = p + q$, and $n = y + z$, then $mn = py + pz + qy + qz$.
5. If $m = x + y + z$, then $\frac{m}{n} = \frac{x}{n} + \frac{y}{n} + \frac{z}{n}$.
6. If $z = xy$, then $\frac{m}{z} = \frac{m \div x}{y}$.
7. If $pq = z$, then $p \times mq = mz$, and $p \times \frac{q}{m} = \frac{z}{m}$.
8. $pq = mp \times \frac{q}{m}$.
9. If $cd = ef$, then $e = \frac{cd}{f}$, $c = \frac{ef}{d}$, $d = \frac{ef}{c}$, $f = \frac{cd}{e}$,
10. If $\frac{m}{n} = o$, then $\frac{xm}{n} = xo$, and $\frac{m \div a}{n} = o \div a$.
11. If $\frac{m}{n} = o$, then $\frac{m}{pn} = \frac{o}{p}$, and $\frac{m}{n \div p} = op$.
12. $\frac{m}{n} = \frac{mx}{nx} = \frac{m \div x}{n \div x}$

EXERCISE XLVI.

MISCELLANEOUS EXERCISES.

1. The less of two numbers is 347 and their difference 58, what is their product?

2. Find one number equivalent to each of the following expressions:—

$$\frac{289+735}{18}; \quad \frac{587\times42}{609-14}; \quad (237\times14\times8)\div(19+7-6).$$

3. What is the price of 1 yard when 48 cost £15 10s. 4d.?

4. Multiply 2s. 4d. by 215; and 3s. 9d. by 175.

5. What will be the amount of the wages of 6 labourers for 28½ days at 2s. 3d. each per day?

6. Divide £3 15s. 5½d. by 23; and £50 by 17.

7. Find the product of the sum and difference of the two numbers, 792 and 68.

8. What is the difference between the sum of the squares of 519 and 432 and the square of their sum?

9. Simplify the following expressions:—
   (a). $128+19+64-(8+123)$.
   (b). $796-14+8-3+153$.
   (c). $2740\times(58+6)$; $3709\times695+8$.
   (d). $2718-(45+6-1)$.
   (e). $(409-8)\times(627+14)$; $(5296+13)\div(20-3)$.

10. How many times will a wheel 7 ft. 3 in. in circumference revolve in traversing 14 miles?

11. How many farthings are there in 27 sovereigns, 50 half-crowns, 87 shillings, and 43 sixpences?

12. How many minutes are there in 365 days, 5 hours, and 48 minutes?

13. In £58 12s. 6d. how many francs at 10d. each?

14. Reduce 125 yds. 2 ft. 4 in. to inches.

15. When an English sovereign is exchanged in Belgium for 25 francs 20 centimes (100 centimes = 1 franc); in Prussia for 6 thalers 20 groschen (30 groschen = 1 thaler); and in Frankfort for 12 florins (60 kreutzers = 1 florin); how would the sum of £12 14s. 6d. (English) be paid in the money of each of these countries?

16. Reduce £123 15s 9¾d. to farthings.

17. How many American dollars, value 4s. 2d., are equal in value to £20?

18. How many coins worth 4s. 9d. are there in £231 16s.?

19. How many parcels, each containing 4¼ oz., can be made up out of 3 qrs. 17 lbs.?

20. If 17 men reap 19 acres 2 roods 17 poles in a day, and 8 of them reap one-third of an acre each, how much ought each of the others to reap?

21. A silversmith makes 18 spoons, each weighing 2 oz. 7 dwt. 19 grs ; three dozen and a half others, weighing 1 oz. 11 dwt. 7 grs. ; and 19 silver forks, each weighing 1¾ oz. : how much silver must he use ?

22. What is the worth of 17 lbs. 6 oz. 10 dwt. of gold, at £3 17s. 10½d. per oz. ?

23. If one-thirteenth of a certain gold coinage be alloy, what is the quantity of pure gold in 274 pieces weighing 54 grs. each ?

24. An English sovereign weighs 123 grains ; what is the weight of £235 10s. in gold ?

25. If a man advances 2 ft. 6 in. each step, takes 72 steps per minute, and walks 4½ hours, how far will he go ?

26. Find the difference in feet between the polar and the equatorial diameters of the earth, the one being 7,899 miles 1 fur., and the other 7,925 miles 5 fur.

27. How much English money is equivalent to 187 Italian ducats, at 3s. 1½d. each ?

28. If a plank be 6¾ in. wide, what length of it will give a surface of 2 square feet ?

29. How many pieces worth 2¼d. each are there in 150 guineas, £70, 34 crowns, 17 half-crowns, and 89 sixpences ?

30. A ship is worth £3,700, and the cargo is worth 6 times the ship ; what is the worth of one-fifteenth of the ship and cargo together ?

31. A pack of wool weighing 2 cwt. 1 qr. 19 lbs. costs £15 4s. 10½d., what is its cost per lb. ?

32. A gentleman's yearly income is 900 guineas ; he spends one-twelfth of his income in charity, and spends on an average £11 5s. per week : what can he save in a year ?

33. Three persons purchase a ship worth £24,000 ; the first takes two parts ; the second, three ; and the third, four parts : what is the value of each man's share ?

34. What number subtracted from the square of 29 will leave the product of 16 and 19 ?

35. If a steam-vessel reach a port 3,050 miles distant in 4 weeks 3 days, what is her average speed per hour ?

36. What is the difference between the daily income of a man whose salary is £250 a year and of one who receives £720 per annum ?

37. What is the number which, if the third power of 35 be deducted from it, will leave the second power of 79 as the remainder?

38. What is the total weight of silver in half a dozen dishes, each weighing 49 oz. 3 dwt. 4 grs. ; a dozen plates, each 16 oz. 17 dwt. ; and a salver weighing 126 oz. 15 dwt. 18 grs. ?

39. By how much must the square of the sum of 12 and 3 be multiplied in order to give the third power of the difference between 108 and 59 ?

40. What is the total length of 49 pieces of cloth, each measuring 27 yards 2 qrs. 2 nails ?

41. If I purchase 17 hogsheads weighing 14 cwt. each, at £24 per cwt., at what price must I sell the whole in order to gain 2½d. per lb. ?

42. Suppose in London 1 person dies per annum out of every 44, in a manufacturing town 1 in 41, and in a rural village 1 in 49; how many persons die in one year in all three, assuming the population of the first to be 4,100,000, the second 783,000, and the third 2,793?

43. How many acres are contained in three countries, of which the first comprises 723,100 square miles, the second 12,342, and the third 89,704 square miles?

44. What number, divided by the sum of the cubes of 3 and 4, will give as quotient the square of the sum of 7 and 8?

45. If I buy 1,874 yards of cloth at 4s. 6½d. per yard, and sell at 5s 3d., what do I gain?

46. If a merchant insures his warehouse when stocked at £17,530, but when empty at £2,935, by how much does he calculate that the worth of his stock exceeds that of the warehouse?

47. If 1,792 persons, 292 carriages, and 17 single horses pass through a toll-gate in one day, the first paying one halfpenny, the second 2½d., and the third 1½d. each, how much money is received?

48. How many guineas, sovereigns, crowns, half-crowns, and shillings, and of each an equal number, are there in £1,546 17s. 6d.?

49. If I buy 1,453 gallons of spirits at 6s. 7d. per gallon, and after losing a quantity of it by an accident, gain £25 profit by selling the remainder at 8s. 10d. per gallon, how much was lost?

50. If I have to measure a distance of three furlongs with a line three rods and a half long, how many times will the line measure the distance?

51. How many pounds of tea at 5s. 6d per lb. must be given in exchange for 293 yards of silk at 3s. 4½d. per yard?

52. If 2 cwt. 1 qr. of sugar, costing £3 5s. per cwt., and 1½ cwt. at 4½ guineas, be mixed together, what is the value of a pound of the mixture?

53. A person buys a hogshead of wine in bond for £36; the duty is 5s. 6d. per gallon: what must it be sold at per dozen to gain £15, six bottles being equal to one gallon?

54. If a person gives £46 10s. for 107 gallons, how much water must he add to it in order to reduce its value to 7s. 9d. per gallon?

55. How long would it take to count a billion, supposing 180 could be counted every second, and the counting was to go on 12 hours every day?

56. What does a person pay as his poor-rate whose house is rated at £86, the rate being 10½d. in the pound?

57. How many days old was a person on the 14th of July, 1854, who was born on the 27th of September, 1831?

58. What is a gentleman's income who pays £36 9s. 2d. income tax, when the rate is 7d. in the pound?

59. If a person makes a journey of 500 miles in 16 days, what is the average rate at which he travels per hour, allowing nine hours' walking per day?

PRIME AND COMPOSITE NUMBERS.

SECTION I.—MEASURES AND MULTIPLES.*

137. A number is a *measure* of another, or is said to measure it, when it is contained an exact number of times in that other.

Thus: 5 is a measure of 30, 11 of 55, 12 of 36.

138. A number is called a *multiple* of another when it contains that other an exact number of times.

Thus: 48 is a multiple of 6, 24 of 8, and 56 of 7.

139. *Observation.*—The word "measure" is the correlative of "multiple." Whenever there is a measure, the number of which it is a measure is its multiple, and whenever there is a multiple, the number of which it is a multiple is its measure.

Thus, because 27 is a multiple of 9, 9 is a measure of 27.
So, if x is a measure of y, y is a multiple of x.

140. A number is called a *common measure* † of two or more others when it is a measure of each of them.

Thus: 7 is a common measure of 21, of 70, and of 14.
If a is a measure of b, of c, and of d, it is their common measure.

141. A number is called a *common multiple* of two or more others when it is a multiple of each of them.

Thus, because 48 contains an exact number of sixes, of twelves, of eights, it is a common multiple of 6, 8, and 12.

142. *Observation.*—The product of any two numbers is always their common multiple.

Thus 5 × 6 evidently contains an exact number both of sixes and of fives, and is therefore their common multiple.

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* These words, "measure and multiple," are *relative* terms; they show the relationship between one number and another, and do not describe any abstract property of either. Thus we cannot simply say 12 is a multiple. But 12 is a multiple *of three* or *of four*. So it is unmeaning to say that 9 is a measure merely, unless we say it is a measure *of* some other number, as that 9 is a measure of 27 or of 108.

† The word "*common*" can never be properly used except when a number is considered in relation to two or more others. It is never right, for instance, to say that a is a common measure or a common multiple of b. If it be common it must be common *to two* or more others, as a is a common measure of b and c.

143. Every number is either Prime or Composite.*

A PRIME number is one which has no measure. All other numbers are COMPOSITE.

Thus: 5, 7, 11, 19, are prime numbers.

144. Numbers are said to be prime to one another when they have no common measure. Such numbers are called incommensurable.

Thus, the number 20 is not a prime number absolutely, or considered by itself, nor is the number 9, because both have measures; but as neither of the measures of the one is also a measure of the other, they are *relatively* prime, or are prime to one another, that is to say, they are incommensurable except by unity.

145. *Observation.*—The number of measures a given number may have is limited: it may have none at all, or it may have only one or two, and in all cases the number of measures can easily be determined; but the number of multiples a number may have is unlimited. For we may add a number to itself an infinite number of times, and each time we do this we have a new multiple of it. So also if we take two numbers at random, it may be that they have no common measure; and if they have, the number of such measures will probably be very small and can soon be determined; but they will certainly have a common multiple, and there is no limit to the number of the common multiples which may be taken.

EXERCISE XLVII.

Distinguish by simple inspection the prime from the composite numbers in the following :—

6, 19, 54, 27, 35, 109, 70, 141, 23.
71, 85, 96, 42, 57, 83, 65, 102, 28.

146. AXIOM XVI.—*If one number measures another, it measures all multiples of that other.*

Demonstrative Example.—For if 7 measures 28, or is contained a certain number of times in 28, it must also measure twice 28, or three times 28, or any multiple of 28.

General Formula.—If a measures b it also measures xb.

For let it be granted that a is contained n times in b,

then $b = na$; then $xb = xna$, and contains a xn times.

But a number contained xn times in another measures that other.

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* Prime, from *primus*, first; composite, from *compono*, to place together; *compositus*, put together.

147. AXIOM XVII.—*If one number measures two others, it also measures their sum.*

Demonstrative Example.—Because 6 is a measure of 12 and also of 18, it must also measure 12 + 18, or 30.

General Formula.—If a measures b and c, it measures $b+c$.

For if it measures b it is contained an exact number of times in it; let it be contained n times, then $an = b$. Similarly, let c contain a, m times, then $c = am$: but because $b = an$ and $c = am$, therefore (71) $b+c = a \times (m+n)$; or $b+c$ contains a, $(m+n)$ times, and is therefore a multiple of a.

148. AXIOM XVIII.—*If one number measures two others it measures their difference.*

Demonstrative Example.—Because 5 measures 60 and 15 it must also measure 60 — 15, or 45.

General Formula.—If a measures b and c it measures $b - c$.

As in the last example, let b contain a, n times, and c, m times; then $b - c$ must contain a, $(n - m)$ times, and is therefore a multiple of a.

149. *If one number measures the divisor and dividend in a Division sum it also measures the remainder.*

$$\begin{array}{rrl} 24) & 372 & (15 \\ & \underline{24} & \\ & 132 & \\ & \underline{120} & \\ & 12 & \end{array}$$

Demonstrative Example.—Here let it be granted that 6 is a measure both of 24 and 372, the divisor and dividend: it also measures the remainder, 12.

If n *be a measure of* a *and* b, *the divisor and dividend, it measures* d *the remainder.*

For let the sum be—

$$\begin{array}{rcl} a) & b & (c \\ & \underline{ca} & \\ & d & \end{array}$$

Because n measures a (hypothesis), it measures ca by (146); but it also measures b (hypothesis). Wherefore (148) it also measures d, which is the difference between ca and b.

150. *If one number measures the divisor and remainder it must also measure the dividend.*

Demonstrative Example.—In the last example, because 6 measures 12 and 24 it must also measure 372, for this is the sum of 12 and of a multiple of 24.

General Formula.—If n measures both a the divisor, and d the remainder, it measures b the dividend.

$$\begin{array}{r} a)\ b\ (c \\ ca \\ \hline d \end{array}$$

For n being a measure of a is (146) also a measure of ca; but by hypothesis it also measures d, therefore (147) it must measure the sum of ca and d. But b is the sum of ca and d. Wherefore n measures b.

151. *Corollary.—The greatest number which measures the divisor and remainder is also the greatest which will measure the divisor and dividend.*

Because every number which measures the divisor and remainder (150) also measures the dividend, and because every number which measures divisor and dividend (149) also measures remainder, the greatest which is common to divisor and remainder is also the greatest common to dividend and divisor.

GREATEST COMMON MEASURE.

152. The *greatest common measure* of two or more numbers is the greatest number which will divide them both without a remainder. Whenever the numbers are small, or are within the range of the tables which we know by heart, we may find their greatest measure by simple inspection and without trouble. But whenever the numbers are large we adopt the principle just enunciated, and by making one of the given numbers the divisor, and the other the dividend, we solve the question in the following manner:—

$$\begin{array}{rrrl} 252) & 2097 & (8 & \\ & 2016 & & \\ \cline{2-2} & 81) & 252 & (3 \\ & & 243 & \\ \cline{3-3} & & 9) & 81\ (9 \\ & & & 81 \\ \cline{4-4} \end{array}$$

Example I.—Find the greatest common measure of 252 and 2097. These numbers are too large to be treated by simple inspection, so we divide one by another. Now by (151) we know that the greatest common measure of 2097 and 252, the dividend and divisor, will also be the greatest of 252 and 81, the divisor and remainder. But these are also too large. We make one of them a dividend, and the other the divisor; and again the greatest common measure of this divisor (81) and the new dividend, 252, must also be the greatest common measure of 81 and of 9, the divisor and new remainder. But on applying one of these to the other we find that it is a measure of it, and must therefore be the greatest common measure of both. 9 is therefore the greatest common measure of 252 and 2097.

153. *Example II.*—Let it be required to find the greatest common measure of any two numbers, m and n, and let n be the greater. Let it contain m, 3 times, and leave a remainder o. Let o be contained in m, 5 times, and leave a remainder p, &c., &c.; and let s the last remainder be contained exactly twice in r. *Then* s *is a common measure of* m *and* n.

$$\begin{array}{l}
m)\;n\;(3\\
\;\;\underline{3m}\\
\qquad o)\;m\;(5\\
\qquad\quad\;\underline{5o}\\
\qquad\qquad p)\qquad(8\\
\qquad\qquad\quad\;\;\underline{8p}\\
\qquad\qquad\qquad\; q)\;p\;(2\\
\qquad\qquad\qquad\qquad\underline{2q}\\
\qquad\qquad\qquad\qquad r)\;q\;(1\\
\qquad\qquad\qquad\qquad\qquad\underline{r}\\
\qquad\qquad\qquad\qquad\qquad s)\;r\;(2\\
\qquad\qquad\qquad\qquad\qquad\qquad\underline{2s}
\end{array}$$

For since it measures r it is a common measure of itself and r. But because (150) whatever measures remainder and divisor measures also the dividend, therefore s measures q. Again, for the same reason, since it measures q and r it also measures p, and because it measures p and q (the divisor and remainder), it must also measure o (the dividend). But whatever measures o and p must also measure m, and whatever measures m and o must measure n. But s measures m and o, therefore it measures m and n.

But s *is also the* GREATEST *common measure of* m *and* n.

For, if it be possible, let them have a greater, and let it be a. Then because, according to this hypothesis, a measures m and n, it also measures o (149). And for the same reason, if it measures m and o, the divisor and dividend, a also measures p, the remainder; in like manner it also measures q; and because it measures p and q it also measures the remainder r, and therefore also measures s. But by hypothesis a is greater than s, therefore it cannot measure s, and therefore no greater number than s is a common measure of m and n.*

154. *Observation.*—The object of this process is gradually to diminish the numbers under inspection until they are small enough to have their measure easily ascertained by the tables. We start with two numbers which are too large to allow us to determine their greatest common measure by simple inspection. But as the remainder in a Division sum must always be less than either divisor or dividend, and as whatever is the greatest common measure of the two original numbers is also the greatest common measure of divisor and remainder, the effect of the first

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* Euclid, book vii.

step is to give us two smaller numbers to compare. If these two (the divisor and remainder) are not small enough, we treat them in the same way and find another remainder, which is of course smaller than either. Thus we proceed until we obtain two numbers so small that we can readily tell whether they have or have not a common divisor. If they have, that number is also the common divisor of the two original numbers; if they have not, the two original numbers have not any common measure.

If in the course of the operation we perceive that a remainder, being a prime number, does not measure the preceding remainder, we may at once conclude that the two given numbers are prime to each other; or,

If any two consecutive remainders are observed to have no common measure, it is useless to proceed further, because in that case it will be evident that the numbers themselves have no common measure.

RULE.

155. Divide the greater by the less. If there be no remainder the less of the two numbers is a measure of the greater, and therefore their greatest common measure.

But if there be a remainder, bring down the former divisor and divide it by the first remainder. Afterwards bring down the second divisor and divide it by the second remainder, and so on until there is no remainder. The last divisor is the greatest common measure of the two original numbers.

If the last divisor be 1, the numbers have no common measure, *i. e.*, they are incommensurable.

EXERCISE XLVIII.

☞ Find the greatest common measure of the following numbers:—

1. 35 and 40; 288 and 36; 570 and 930.
2. 266 and 637; 1793 and 462; 125 and 360.
3. 472 and 720; 162 and 2763; 300 and 468.
4. 65935 and 47355; 7228 and 4196; 1286 and 907.
5. 10987 and 1495; 1271 and 31628; 4096 and 84.
6. 4058 and 432; 323 and 1700; 17962 and 815.
7. 13581 and 1836; 1917 and 459; 2308 and 584.
8. 6156 and 1558; 3776 and 1824; 12408 and 8580.
9. 1953 and 13671; 19776 and 5562; 10568 and 9247.
10. 2943 and 8829; 602 and 1032; 4279 and 39678.

TO FIND THE GREATEST COMMON MEASURE OF THREE OR MORE NUMBERS.

156. Suppose it is required to find the greatest common measure of 30, 18, and 21. By (155) we find that the greatest common measure of 30 and 18 is 6, or that whatever measures 30 and 18 must also measure 6. Wherefore whatever measures 30 and 18 and 21 must also measure 6 and 21; and 3, which is the greatest common measure of 6 and 21, must be the greatest common measure of 30, 18, and 21. It is evident that if any two of the given numbers have no common divisor, the whole of the given numbers have no common divisor, and are therefore incommensurable.

To find the greatest common measure of *a*, *b*, *c*, and *d*. Let the greatest common measure of *a* and *b* be *m*; then, because whatever measures *a* and *b* also measures *m*, the greatest common measure of *a*, *b*, and *c* must be that of *m* and *c*; let this be *n*; then because whatever measures *a*, *b*, and *c* also measures *n*, the greatest common measure of *a*, *b*, *c*, and *d* must be that of *n* and *d*; let this be *p*; then *p* is the greatest common measure of *a*, *b*, *c*, and *d*.

RULE.

157. Find by (155) the greatest common divisor of any two of the given numbers. Then find the greatest common measure of the divisor thus obtained, and another of the given numbers; proceed in this way until the numbers are exhausted. The last of these common divisors will be the number sought.

EXERCISE XLIX.

☞ Find the greatest common measure of the following numbers:—

1. 25, 75, and 100; 24, 16, and 80; 64, 48, and 120.
2. 805, 2622, and 1978; 6914, 396, and 5784; 170, 262, 568.
3. 63, 700, and 371; 108, 6144, and 1116; 729, 1371, and 1695.
4. 84, 48, 132, and 396; 568, 744, 64, 32, and 192.
5. 396, 693, 543, and 999; 128, 484, and 5256.
6. 4654, 3968, and 112; 540, 8375, 75, and 1110.
7. 126, 217, and 175; 477, 1629, 666, and 3726.

158. AXIOM XIX.—*Every number is either a prime number or may be resolved into prime factors.*

Demonstrative Example.—For if the number can be divided by another it is not a prime number, and is therefore measured by the divisor and the quotient, which are its factors. Either of these factors which has a measure may also be resolved into its factors, and this process may evidently be carried on until all the factors are prime numbers.

Let a be a number: if it has no measure it is a prime number; but if it has, let it equal $b \times c$; then if b and c are prime numbers a is resolved into its prime factors; if not, let $b = e \times f$, and $c = g \times h$, and let e, f, g, h be prime numbers. Then $a = e \times f \times g \times h$, and is resolved into its prime factors.

RULE TO RESOLVE A NUMBER INTO ITS PRIME FACTORS.

159. Divide by the smallest prime number which will measure it. Then divide the quotient so found by the smallest prime number which it contains; and proceed in this way until a quotient occurs which cannot be divided. The series of divisors and the last quotient are the prime factors.

Example.—Resolve 390 into its prime factors. Here we divide first by 2, then by 3, then by 5, and at last come to a quotient which is a prime number. The number is thus found to consist of 2 × 3 × 5 × 13. But by the rule, each of the numbers thus chosen was a prime number. Wherefore 390 has been resolved into its prime factors.

2)390
3)195
5)65
13

Ans. $390 = 2 \times 3 \times 5 \times 13$.

EXERCISE L.

☞ Resolve each of the following numbers into its prime factors:—

1. 347; 58; 196.
2. 7189; 6541; 4127.
3. 1027; 3526; 4098.
4. 23169; 5481; 71086.
5. 3851; 2046; 4578.
6. 2169; 540; 770.
7. 4096; 225; 1836.
8. 3062; 1008; 8152.

160. It is not difficult to ascertain how many prime numbers can be found within any given limit. If we set down a column of all the numbers in order, from 1 to 100, we may mark off those which are not prime numbers in the following manner:—

Begin by marking off every number which contains two, that being he lowest prime number; thus 4, 6, 8, and the whole series of even numbers will be marked. It is evident that not one of these is a prime number, as every one is divisible by 2. Then commence from 3, which is the next prime number, marking off every number which contains 3. Thus 6, 9, 12, 15, and the whole series of multiples of 3 will be excluded from the list of prime numbers. In like manner commence from 5 (the prime number next in order), and point off every number in the series which contains an exact number of fives. Thus 10, 15, 20, &c., will be excluded from the list. Again, take 7 (the next prime number), and mark off its multiples 14, 21, 28, &c. ; then do the same with 11, and so on with all the prime numbers in succession. Every number which is not prime will be excluded. In the following table the number of asterisks affixed to each figure shows the number of its prime factors. In the series of numbers given it will be seen that the only prime numbers are—2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.*

1	
2	
3	
4	*
5	
6	* *
7	
8	*
9	*
10	* *
11	
12	* *
13	
14	* *
15	* *
16	*
17	
18	* *
19	
20	* *
21	* *
22	* *
23	
24	* *
25	*
26	* *
27	*
28	* *
29	
30	* * *
31	
32	*
33	* *
34	* *
35	* *
36	* *
37	
38	* *
39	* *
40	* *
41	
42	* * *
43	
44	* *
45	* *
46	* *
47	
48	* *
49	*
50	* *
51	* *
52	* *

EXERCISE LI.

Make out a similar list as far as two hundred, and ascertain how many prime numbers there are, and how many prime factors each composite number has.

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* This process of eliminating all the composite numbers from a series was invented by Eratosthenes, a Cyrenian Greek, who had charge of the Alexandrian Library in the time of the second Ptolemy, and paid great attention to Mathematical Science. It is called the Sieve of Eratosthenes.

LEAST COMMON MULTIPLE.

161. The smallest number which can be divided by two or more others without a remainder is called the least common multiple of those others.

162. *The product of two or more numbers is a common multiple of all of them.*

Demonstrative Example.—The product of 4, 7, and 9 must contain an exact number of fours, of sevens, and of nines, and is therefore a multiple of each of them. A common multiple of any set of numbers can always therefore be found by multiplying them together.

General Formula.—If $a \times b = c$,
then c is a common multiple of a and b.

163. *If two or more numbers have a common measure, their product divided by that common measure will be a common multiple of those numbers.*

Demonstrative Example.—If it be required to find a common multiple of 30 and 12, we may do it at once (162) by taking their product, for $30 \times 12 = 360$ = a common multiple of 30 and 12. But because 6 is their common measure, and $30 = 6 \times 5$, and $12 = 6 \times 2$, it is evident that $6 \times 2 \times 5$ will be a multiple of both, for it will contain the 12 five times, and the 30 two times.

Therefore 60, or 12×30 divided by their greatest common measure, is the common multiple of 30 and 12.

General Formula.—If $a = mx$ and $b = my$,
then mxy is a common multiple of a and b.

RULE TO FIND THE LEAST COMMON MULTIPLE OF TWO NUMBERS.

164. Find (155) their greatest common measure, and divide their product by this number.

Exercise LII.

☞ Find the least common multiples of the following numbers :—

1. 25 and 35 ; 72 and 108 ; 63 and 99.
2. 54 and 842 ; 300 and 42 ; 75 and 100.
3. 82 and 96 ; 7123 and 456 ; 372 and 48.
4. 235 and 195 ; 124 and 16 ; 50 and 30.

165. *If in any set of numbers two or more are found having a common factor, the product of these numbers divided by the common factors will be a common multiple of all of them.*

Demonstrative Example.—Find the least common multiple of the four numbers, 16, 20, 21, and 14. Here, because $16 = 4 \times 4$, $20 = 5 \times 4$, $21 = 3 \times 7$, $14 = 2 \times 7$; therefore $4 \times 4 \times 5 \times 7 \times 3$ will be a multiple of each of them, for it will contain all the factors of each; but this will equal the whole product divided by $4 \times 7 \times 2$.

General Formula.—If $a = mx$, $b = xy$, $c = yz$, and $d = mz$,

then $mxyz$ is a common multiple of a, b, c, and d;

for it contains a yz times, b mz times, c mx times, d xy times.

166. The exclusion of common factors is usually effected by the following process :—

Find the least common multiple of 15, 18, 20, 32, 12, and 100.

$$\begin{array}{r|rrrrrr}
3 & 15 & 18 & 20 & 32 & 12 & 100 \\ \hline
4 & 5 & 6 & 20 & 32 & 4 & 100 \\ \hline
5 & 5 & 6 & 5 & 8 & 1 & 25 \\ \hline
2 & 1 & 6 & 1 & 8 & 1 & 5 \\ \hline
 & & 3 & 1 & 4 & 1 & 5
\end{array}$$

$3 \times 4 \times 5 \times 2 \times 3 \times 4 \times 5 = 7200$, the least common multiple.

Here it was first observed that 3 was a common measure of several of the given numbers, viz., 15, 18, and 12. On dividing we find that (165) $3 \times 5 \times 6 \times 4$ will contain each of them. The other numbers, 20, 32, and 100 are brought down.

At the second line we have this result :—

The product of $3 \times 5 \times 6 \times 20 \times 32 \times 4 \times 100$ is a multiple of each of the original numbers.

But three of these numbers, 20, 32, and 100, are divisible by 4.

Hence $4 \times 5 \times 8 \times 25$ is a common multiple of $20 \times 32 \times 100$.

At the third line we have this result :—

$3 \times 4 \times 5 \times 6 \times 5 \times 8 \times 25$ is a multiple of each of the original numbers.

Again, dividing these numbers by 5 and by 2, we have—

The product of 3, 4, 5, 2, 3, 4, and 5, or 7200, is a common multiple of all the original numbers. This is evidently a much smaller number than the product of 15, 18, 20, 32, 12, and 100.

167. *Observation.*—The division by common factors in any order will always give a *less* common multiple than the product of the given numbers, but it does not always give their *least* common multiple. In order to obtain this it is necessary only to divide by prime numbers, or by a number which, like 4, is the square of a prime number.

168. The same result might have been obtained by resolving the numbers into their prime factors by (159). Thus:—

15=3 × 5
18=3 × 3 × 2
20=2 × 2 × 5
32=2 × 2 × 2 × 2 × 2
12=3 × 2 × 2
100=5 × 5 × 2 × 2

Because 2 occurs five times in one of these numbers (in 32), therefore if we take the product of 5 twos, or the fifth power of 2, as it is called, in our common multiple all the other twos may be neglected; and because the highest power of 3 which occurs is the 2nd (in 18), therefore if we take 3 × 3 in the common multiple the other threes may be neglected. Similarly, because 5 × 5 occurs in the last number (in 100) the fives which occur in the other numbers may be neglected. Hence, 2 × 2 × 2 × 2 × 2 × 3 × 3 × 5 × 5 = 7200; and this is the least common multiple of 15, 18, 20, 32, 12, and 100, for it contains all the prime factors of each of them.

But the product of all the numbers is the product of 12 twos, 4 threes, and 4 fives, which is a much larger number than 7200.

TO FIND THE LEAST COMMON MULTIPLE OF THREE OR MORE NUMBERS.

RULE I.

169. Divide as many of them as possible by any prime number which is a common measure of any of them. Bring down the undivided numbers, and divide those in the second line by any other prime measure of any of them. Proceed in this way until no common measure can be found. The product of all the divisors and of the numbers remaining in the lowest line will be the least common multiple required.

RULE II.

170. Resolve all the given numbers into their prime factors. Take each of these factors to the highest power which occurs in any one of the given numbers. The product will be the least common multiple.

Observation.—It is evident that if any one of the numbers is seen to be contained in another it may be struck out at once. Thus, in the example given, because 20 is a measure of 100, whatever is a multiple of 100 will certainly be a multiple of 20, and therefore if a multiple of the 100 be found the 20 may be neglected.

EXERCISE LIII.

☞ Find the least common multiple of the following numbers :—
(a) The first four sums by the second method, and
(b) The whole by the first.

1. 25, 75, and 30 ; 12, 28, and 44 ; 29, 16, and 40.
2. 15, 14, 16, and 18 ; 27, 30, 54, and 27.
3. 105, 110, 14, and 17 ; 354, 63, 852, and 81.
4. 1, 2, 3, 4, 5, 6, 7, 8, and 9 ; 20, 12, 15, 18, 4.
5. 2712, 816, 54, and 15 ; 21, 27, 36, and 19.
6. 5908, 5612, and 3047 ; 250, 360, 49, and 700.
7. 162, 108, 81, and 54 ; 608, 1344, 160, 352, and 96.
8. 913, 581, 2324, 6, and 249 ; 2064, 22704, 6192, and 14448.
9. 8468, 6351, 19053, and 14819 ; 9217, 1418, 4963, and 709.
10. 4004, 2548, 728, and 4732 ; 259, 222, 74, and 185.
11. 209, 133, 95, and 57; 58, 81, 153, and 42.

SECTION II.—SPECIAL PROPERTIES OF NUMBERS.

For convenience in working subsequent rules, and especially in determining the common factors of numbers by simple inspection, the following truths are important.

171. *Every even number is divisible by* 2, *and every number whose last two digits are divisible by* 4 *is itself divisible by* 4.

The former assertion is self-evident ; and because 100 = 25 × 4, therefore any number of hundreds is a multiple of 4 ; and if the two last digits also form a multiple of 4, the whole number is divisible by that number ; *e.g.*, 1800, 16924, 24936, are divisible by 4.

172. *Every number whose last three digits are divisible by* 8 *is itself divisible by* 8.

For because 1000 is divisible by 8 any number of thousands is so likewise. Hence if the three last digits form a multiple of 8 the whole is a multiple of 8; *e. g.*, 17000, 25328, and 491720 are divisible by 8.

173. *Every number ending in* 0 *or* 5 *is divisible by* 5.

For every number ending in 0 consists of an exact number of tens, and 10 = 5 × 2. Again, every number ending in 5 consists of a number of tens + 5, and is therefore a multiple of 5.

174. *Every prime number of two or more digits terminates with one or other of the digits* 1, 3, 7, *or* 9.

For numbers terminating in any other digit have already been shown to be composite.

175. *Every number the sum of whose digits is divisible by* 9 *or* 3 *is itself divisible by* 9 *or* 3, *and every number the sum of whose digits if divided by* 9 *or* 3 *would give a certain remainder, would itself if divided by* 9 *or* 3 *give the same remainder.*

For because 10 = 9 + 1, 7 tens must contain 7 nines + 7, 3 tens = 3 nines + 3, and if any number of tens be taken, it will, if divided by 9, give the same number as a remainder; *e. g.*, *n* tens ÷ 9 = *n* times and *n* remainder.

So also, because 100 = 99 + 1 and 99 = 11 × 9; therefore 700 = (7 × 99) + 7, and *n* hundreds divided by 9 will give *n* remainder. The same is also true of thousands, tens of thousands, millions, &c. Suppose, for example, it is required to divide 723425 by 9.

$$\text{Now by (93) } \frac{723425}{9} = \frac{700000}{9} + \frac{20000}{9} + \frac{3000}{9} + \frac{400}{9} + \frac{20}{9} + \frac{5}{9}.$$

If the division be performed separately on each of these parts we shall have a series of remainders, 7, 2, 3, 4, 2, and 5. If the sum of these remainders be divisible by 9 the whole number is divisible by 9, but not otherwise. Here the sum of the digits is 23, or (2 × 9) + 5; therefore, if the whole number were divided by 9 the remainder would be 5.

Because 3 is a measure of 9, 10 and every power of 10, if diminished by 1, would be a multiple of 3. Hence, if the sum of the digits composing a number be divisible by 3 the number itself is divisible by 3.

The method of proving the four simple rules by casting out the nines is founded on this truth. For if one factor divided by 9 give a certain remainder, and another factor divided by it give a second remainder, then the product of these two factors divided by 9 will give either the product of these remainders or the same remainder as that product. Hence, if we find by casting out the nines of the multiplicand and multiplier, separately, that the product of these two remainders divided by 9 gives the same remainder as is found by casting out the nines of the answer, that answer is likely to be right; although it is evident that if the answer contained exactly 9 too much or too little, or if a cipher were incorrectly placed in the answer, this method would fail to detect the error.

The method of proving Division will appear exactly the same when it is remembered that the dividend is always the product of the two factors, divisor and quotient.

Observation.—This peculiarity of the number 9 is not an essential property of the number itself, but is a consequence of our having adopted a decimal notation. Had the base of our system of Arithmetic been 5 instead of 10, the same property would have belonged to the number 4; and generally, if r represent the scale of notation, the sum of the digits divided by $r - 1$ always gives the same remainder as if the whole number were divided by it.[*]

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[*] Many of the results of the principle here explained are rather curious than useful, although it is a good exercise for learners to find out similar peculiarities and to trace out their reasons. For example—

If in any line of figures the sum of the digits be subtracted from the whole, the remainder is always a multiple of 9; *and if the figures composing a line be transposed in any order and subtracted from the line itself, the remainder is also divisible by nine.*

Example I. 723425 = (a multiple of 9) + 5
Subtract 5
723420 a multiple of 9

II. 723425 = (a multiple of 9) + 5
Subtract 23 = the sum of the digits, or (2 × 9) + 5
723402 = a multiple of 9

III. 723425 = (a multiple of 9) + 5
Subtract the same figs. transposed 437522 = (a multiple of 9) + 5
285903 = a multiple of 9.

It is evident that having once ascertained that the original number if divided by 9 leaves a remainder 5, we may take away either 5, or 9 + 5, or any number of nines + 5 from that number, and an exact multiple of 9 will be left. But the same figures transposed in any order will always give the same sum; and the number they represent will always therefore be a multiple of 9 + 5.

176. *If an even number have the sum of its digits divisible by* 3, *that number is divisible by* 6.

For because it is even it is divisible by 2, and if it be also divisible by 3 it is divisible by 2 × 3 or 6. Thus 528, 3738, 2346, are divisible by 6.

177. *If the sum of the digits of an even number be divisible by* 9, *the number is divisible by* 18, *and if the sum be divisible by* 3, *while the tens and units are divisible by* 4, 12 *is a measure of the number.*

The proof of this is the same as in the last case.

178. *Every prime number greater than* 6 *would, if increased or diminished by unity, become divisible by* 6.

For every number greater than 6 would, if divided by 6, leave a remainder 1, 2, 3, 4, 5, or 0. If the remainder be 0 it is not a prime number. If the remainder be 2 or 4 the number consists of $6n + 2$ or $6n + 4$, and in either case is divisible by 2. If the remainder be 3 the number consists of $6n + 3$, and is divisible by 3. Therefore every prime number is either $6n + 5$ or $6n + 1$. In the former case adding 1 to it, and in the latter case taking 1 from it, would make it a multiple of 6. Hence any one of the prime numbers 7, 11, 13, 17, 19, 23, 29, 31, 37, &c., would, if increased or diminished by unity, become divisible by 6. It does not however follow that all numbers expressible by $6n + 5$ or by $6n + 1$ are prime.

Questions on Prime and Composite Numbers.

Define the words measure, multiple, prime, composite, common measure, common multiple, incommensurable.

Deduce three inferences from the supposition that a measures b and c. State the axioms relating to this subject. What propositions applicable to every Division sum are founded on these axioms?

Give the rule for finding the greatest common measure of two numbers. Take 343 and 539 as an example, and demonstrate each step of the process. How is the greatest common measure of three or more numbers ascertained? Describe the process of resolving a number into prime factors, and of excluding all the composite numbers from a given series.

In what way can a common multiple of two or more numbers always be found? When will a less number serve the purpose? Give the rule for finding the least common multiple—(1) of two and (2) of three or more numbers.

How may you tell by simple inspection when a number is divisible by 2, by 4, by 8, o by 5? What numbers are divisible by 9 or by 3, by 18 or by 6? By what test may a prime number be known? Show how the method of proving Multiplication or Division by casting out nines is to be demonstrated.

FRACTIONS.[*]

SECTION I.—NOTATION OF VULGAR FRACTIONS.

179. That part of Arithmetic which treats of whole numbers is called Integral Arithmetic, or the Arithmetic of Integers.[†] In Integral Arithmetic unity is taken as the standard, and considered capable of *repetition.* In Fractional Arithmetic unity is taken as the standard, and considered capable of *division.* In the former we are concerned only with magnitudes *greater than one;* in the latter with magnitudes *less than one.*[‡]

The figures 1, 2, 3, 4, 5, &c., all represent different *multiples* of unity ; the expressions which occur in Fractions refer to the *parts* of unity.

180. In (96) and (99) the expressions $\frac{4}{6}$ and $\frac{20}{28}$ were used at the end of Division sums, and stated to mean the *sixth part of four*, and *the twenty-eighth part of twenty*, respectively. We were then not able to do more with such expressions than to leave them as divisions which had not been completed. The object of Fractional Arithmetic is to investigate such expressions more fully, and to give us an extended notion of Division.

181. Every fraction represents a quotient, the upper number being the dividend and the lower the divisor.

182. The lower of the two numbers is called the DENOMINATOR[§] or namer, because it shows *what parts* of unity have to be taken ; and the upper is called the NUMERATOR[§] or numberer, because it shows *how many* of these parts are to be taken.

Thus $\frac{8}{9}$ signifies that unity is divided into nine parts, of which eight are taken ; it also means the ninth part of eight, the part which eight is of nine, and the quotient which would be obtained on dividing eight by nine.

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[*] From *frango*, I break ; *fractus*, broken (Latin).

[†] *Integer* (Latin), whole or undivided.

[‡] *Preliminary Mental Exercise.*—Before commencing this Rule many questions on the Tables involving both Multiplication and Division should be solved mentally. Such exercise may take two forms ; *e.g.*, I. Find $\frac{6}{7}$ of 35 $\frac{7}{12}$ of 96, $\frac{3}{8}$ of 56, $\frac{6}{11}$ of 55, &c. ; and II. What is that number of which 20 is $\frac{5}{7}$; of which 15 is $\frac{3}{8}$; of which 24 is $\frac{12}{18}$; of which 18 is $\frac{9}{11}$? &c., &c. See *School Arithmetic*, p. 77.

[§] (Latin) *numero*, I number or count ; and *denomino*, I give a name to.

183. Fractions are called *proper* when their denominator is greater than their numerator, and *improper* when the numerator is the greater, or when both are equal.

Thus $\frac{8}{13}$ is a Proper Fraction.

But $\frac{9}{5}$ and $\frac{12}{12}$ are Improper Fractions.

The last expression, $\frac{12}{12}$, means that 1 is divided into twelve equal parts, and that twelve of them are to be taken; in other words, $\frac{12}{12}$ means the whole unit; $\frac{13}{12}$ means something more than the whole unit.

Whenever the numerator exceeds the denominator the fraction represents more than one, and as this is not a part of unity merely, but a whole number and a part, the expression is called an improper fraction. Whenever the numerator equals the denominator the expression represents exactly one, and is still called *improper*, because it is not a fraction at all, but a whole number. But whenever the numerator is less than the denominator it signifies that all the parts into which unity has been divided are not to be taken, and therefore that the expression is really fractional. It is hence called a *proper* fraction.

Nevertheless, such expressions as $\frac{11}{10}$ or $\frac{9}{8}$ will often be met with in fractions, and are subject in all respects to the same rules and the same considerations as what are called proper fractions. The distinction therefore is not of much importance, and makes no new rules or statements necessary.

184. A *mixed* number is one which consists of one or more whole numbers and a fraction, as $2\frac{3}{4}$, $5\frac{1}{2}$, $18\frac{7}{15}$.

185. *Every mixed number may have its form altered to that of an improper fraction, and every improper fraction to that of a whole or mixed number.*

Demonstrative Example.—The mixed number $5\frac{7}{8}$ consists of five whole numbers and seven eighths. But because $1=\frac{8}{8}$ therefore 5 must equal $5 \times \frac{8}{8}$ or $\frac{40}{8}$.

$$\text{Hence } 5\tfrac{7}{8}=\tfrac{40}{8}+\tfrac{7}{8} \text{ or } \tfrac{47}{8}.$$

Again, the improper fraction $\frac{29}{6}$ contains more than a whole number. For because $\frac{6}{6}=1$, $\frac{29}{6}$ contains 1 four times and leaves a remainder of 5. There are therefore four whole numbers and five sixths in $\frac{29}{6}$; or $\frac{29}{6}=4\frac{5}{6}$.

General Formula.—$a+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}.$

186. TO REDUCE A MIXED NUMBER TO AN IMPROPER FRACTION—

RULE.

Multiply the whole number by the denominator of the fraction; add the numerator, and place the denominator under the sum.

EXERCISE LIV.

☞ Reduce the following mixed numbers to improper fractions :—

1. $7\frac{5}{12}$; $8\frac{4}{9}$; $3\frac{7}{11}$.
2. $31\frac{4}{7}$; $54\frac{7}{15}$; $23\frac{14}{15}$.
3. $123\frac{4}{9}$; $21\frac{3}{13}$; $1\frac{6}{7}$.
4. $47\frac{1}{8}$; $2\frac{4}{15}$; $183\frac{13}{27}$.
5. $21\frac{3}{7}$; $15\frac{4}{9}$; $m+\frac{a}{n}$.
6. $14\frac{2}{9}$; $28\frac{8}{11}$; $a+\frac{1}{b}$.

187. TO REDUCE AN IMPROPER FRACTION TO A MIXED NUMBER—

RULE.

Divide the numerator by the denominator; the quotient will be a whole number, and the remainder will be the numerator of the fraction.

EXERCISE LV.

☞ Reduce the following fractions to mixed numbers.

1. $\frac{41}{9}$; $\frac{88}{5}$; $\frac{271}{15}$.
2. $\frac{88}{6}$; $\frac{419}{11}$; $\frac{8179}{125}$.
3. $\frac{475}{4}$; $\frac{219}{18}$; $\frac{624}{5}$.
4. $\frac{2105}{17}$; $\frac{8806}{218}$; $\frac{4179}{187}$.
5. $\frac{4198}{263}$; $\frac{6107}{341}$; $\frac{4003}{18}$.
6. $\frac{7296}{81}$; $\frac{312}{75}$; $\frac{4892}{82}$.

188. Every principle which has been demonstrated concerning the relations of dividend, divisor, and quotient (130 *et seq.*), is also true of numerator, denominator, and fraction; *e.g.*—

189. *If the numerator of a fraction be increased or diminished the fraction is increased or diminished in the same degree.*

For whatever be the meaning of $\frac{6}{7}$ the fraction $\frac{12}{7}$ or $\frac{2 \times 6}{7}$ means twice as much, and $\frac{3}{7}$ or $\frac{6 \div 2}{7}$ means only half as much.

190. *If the denominator of a fraction be increased the fraction itself is diminished, but if the denominator*

be diminished the fraction is increased in the same degree.

For whatever be the meaning of $\frac{6}{8}$, the fraction $\frac{6}{16}$ or $\frac{6}{8 \times 2}$ means only half as much, because here the dividend remains the same, and the divisor is doubled, wherefore (131) the quotient is diminished one-half.

191. *Corollary I.—A fraction may be multiplied by a whole number either by multiplying its numerator or by dividing its denominator by that number.*

Example.—$\frac{3}{10} \times 2 = \frac{2 \times 3}{10} = \frac{6}{10}$ or $\frac{3}{10 \div 2} = \frac{3}{5}$.

General Formula.—$\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b \div c}$.

192. *Corollary II.—A fraction may be divided by a whole number either by dividing the numerator or by multiplying the denominator by that number.*

Example.—$\frac{10}{15} \div 5 = \frac{10 \div 5}{15} = \frac{2}{15}$ or $\frac{10}{15 \times 5} = \frac{10}{75}$.

General Formula.—$\frac{a}{b} \div c = \frac{a \div c}{b} = \frac{a}{bc}$.

193. *Observation.*—The method to be employed depends on the character of the numerator or denominator in each case, as seen by simple inspection.

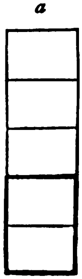
EXERCISE LVI.

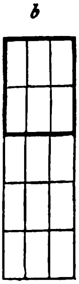
(a) 1. Multiply $\frac{3}{7}$ by 6; $\frac{2}{9}$ by 3; $\frac{7}{12}$ by 4.
2. $\frac{4}{7} \times 6$; $\frac{3}{19} \times 7$; $\frac{8}{15} \times 5$.
3. $\frac{7}{19} \times 4$; $\frac{3}{8} \times 5$; $\frac{47}{60} \times 2$.
4. $\frac{18}{25} \times 5$; $\frac{8}{14} \times 7$; $\frac{13}{34} \times 17$.
5. $\frac{14}{15} \times 3$; $\frac{9}{16} \times 8$; $\frac{14}{27} \times 9$.
6. $\frac{5}{19} \times 17$; $\frac{28}{68} \times 34$; $\frac{58}{91} \times 13$.

(b) 1. Divide $\frac{7}{15}$ by 6; $\frac{8}{14}$ by 5; $\frac{12}{17}$ by 6.
2. $\frac{7}{12} \div 3$; $\frac{18}{29} \div 6$; $\frac{41}{320} \div 2$.
3. $\frac{15}{19} \div 5$; $\frac{16}{47} \div 8$; $\frac{47}{64} \div 5$.
4. $\frac{21}{65} \div 7$; $\frac{28}{75} \div 14$; $\frac{16}{19} \div 7$.
5. $\frac{12}{75} \div 12$; $\frac{4}{19} \div 3$; $\frac{16}{82} \div 4$.
6. $\frac{18}{84} \div 14$; $\frac{28}{60} \div 20$; $\frac{47}{59} \div 10$.

194. *If the numerator and denominator be both multiplied or both divided by the same number, the value of the fraction remains unaltered.**

Demonstrative Example I.—Take the fraction $\frac{6}{15}$. By (189) if the numerator be multiplied by 2 it becomes $\frac{12}{15}$, or twice as much; while by (190) if the denominator be multiplied by 2 the fraction becomes $\frac{6}{30}$, or half as much. But if both be multiplied by 2, and it becomes $\frac{12}{30}$, it is manifest that the one change will neutralize the other, and that the fraction will remain unaltered. Similarly, if both the numerator and denominator of $\frac{6}{15}$ be divided by 3, and it becomes $\frac{2}{5}$, it will have been increased by one process just as much as it is diminished by the other, and will therefore be unaltered.





Demonstrative Example II.—In the diagram (*a*) the dark line surrounds a space which is evidently two-fifths ($\frac{2}{5}$) of the whole surface. In (*b*) the dark line surrounds a space which is the same fraction of the whole but contains six fifteenths ($\frac{6}{15}$). For because in *a* each portion is three times as large as each in *b*, three times as many portions are required in the latter to express the same magnitude. Hence $\frac{2}{5} = \frac{6}{15}$.

General Formula.—$\frac{a}{b} = \frac{an}{bn} = \frac{a + n}{b \div n}$

This truth is the same as that expressed in Axiom XV. (133).

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* It is necessary to notice that equal *additions* to both numerator and denominator make an important difference in the value of the fractions. For example, in the series of fractions :—

$\frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}, \frac{9}{11}, \frac{10}{12}, \frac{11}{13}, \frac{12}{14}, \frac{13}{15}, \frac{14}{16},$

it may be noticed that *one* is added to both numerator and denominator at each successive step. But because the original numerator, 3, is less than the denominator, 5, the increase is *relatively* greater to the numerator than to the denominator, and the fraction is increased at each step. Nevertheless there is a limit to this increase, for equal additions can never make the fraction mean so much as *one*. $\frac{1000003}{1000005}$ would still be less than unity.

If, however, the fraction be improper, as $\frac{9}{4}$, equal additions will have the effect of *diminishing* the fraction; thus in the following series of fractions, which are formed by equal additions of one to both numbers,—

$\frac{9}{4}, \frac{10}{5}, \frac{11}{6}, \frac{12}{7}, \frac{13}{8}, \frac{14}{9}, \frac{15}{10}, \frac{16}{11}, \frac{17}{12}, \frac{18}{13}, \frac{19}{14}, \frac{20}{15},$

every fraction is less than that on its left. In this case the limit of unity is again

195. Every fraction may be expressed in an unlimited number of forms. When it is expressed by higher numbers the new numerator and denominator are formed by equal multiplication, and are called *equi-multiples* of the original figures. When it is expressed in a lower name the numerator and denominator are formed from the first by equal division, and are sometimes called *equi-sub-multiples* of the original numbers.

196. By (145) it will be seen that the number of multiples a given number may have is unlimited, but the number of measures is limited. Hence every fraction may be expressed in an infinite number of ways by using two greater numbers; but it can never be expressed by lower numbers unless the numerator and denominator have a common measure.

EXERCISE LVII.

☞ (a) Express each of the following fractions in four different ways:—

1. $\frac{5}{6}$; $\frac{10}{15}$; $\frac{31}{40}$.
2. $\frac{8}{12}$; $\frac{16}{20}$; $\frac{41}{54}$.
3. $\frac{7}{10}$; $\frac{25}{100}$; $\frac{14}{18}$.
4. $\frac{65}{80}$; $\frac{45}{90}$; $\frac{21}{84}$.

(b) Express each of the following numbers in fractional forms:—

1. 17 with the denominators 3, 8, and 7.
2. 4 .... 5, 9, and 6.
3. 12 .... 8, 3, and 4.
4. 4 .... 15, 12, and 9.
5. 9 .... 21, 3, and 14.
6. 15 .... 44, 116, 2436.
7. 120 .... 480, 666, 568.
8. *a* .... 9, 13, 5.
9. 7 .... *a*, *b*, *c*.
10. *a* .... *b*, *c*, *d*.

---

constantly approached but never reached, for if a million were added to both numerator and denominator, and it became $\frac{1000008}{1000004}$, it would not be so small as one. $\frac{a+n}{b+n}$ is therefore greater or less than $\frac{a}{b}$, according as $\frac{a}{b}$ is a proper or improper fraction, while $\frac{a-n}{b-n}$ is less than $\frac{a}{o}$ if that fraction is proper, but greater if it is improper.

TO REDUCE A FRACTION TO ITS LOWEST NAME.

197. It is often convenient to express a given fraction in its lowest terms. Whenever the fraction is thus expressed the numerator and denominator must be prime to one another, for otherwise they could be divided by their common measure.

RULE.

198. Find by (155) the greatest common measure of the numerator and denominator. Divide both by that measure. The resulting fraction will be equal to the first and will be expressed in its lowest terms.

Example.—Reduce $\frac{252}{1084}$ to its lowest terms.

```
252) 1084 (4
     1008
     ----
       76) 252 (3
           228
           ---
            24) 76 (3
                72
                --
                 4) 24 (6
                    24
```

Here by (155) 4 is found to be the greatest common measure of 252 and 1084. Hence—

$$\frac{252}{1084} = \frac{252 \div 4}{1084 \div 4} = \frac{63}{271}.$$

EXERCISE LVIII.

☞ Reduce the following fractions to their lowest names:—*

1. $\frac{175}{215}$; $\frac{110}{[illegible]}$; $\frac{18}{24}$.
2. $\frac{228}{836}$; $\frac{60}{135}$; $\frac{1100}{8725}$.
3. $\frac{615}{822}$; $\frac{270}{[illegible]}$; $\frac{784}{791}$.
4. $\frac{330}{684}$; $\frac{1184}{1360}$; $\frac{1004}{1056}$.
5. $\frac{532}{931}$; $\frac{192}{5120}$; $\frac{1428}{2652}$.
6. $\frac{205}{6560}$; $\frac{2271}{8466}$; $\frac{396}{1485}$.
7. $\frac{2124}{3717}$; $\frac{48}{69}$; $\frac{1248}{6656}$.
8. $\frac{9324}{14274}$; $\frac{1755}{2241}$; $\frac{12234}{30426}$.
9. $\frac{728}{1092}$; $\frac{10648}{12584}$; $\frac{47481}{74613}$.
10. $\frac{3921}{24838}$; $\frac{6664}{7448}$; $\frac{29061}{38289}$.

---

* In many cases this may be done by simple inspection. See section on Special Properties of Numbers, p. 104.

199. In order to add, subtract, or in any way compare fractions, it is always necessary to reduce them to a common denominator.

If two fractions, $\frac{5}{9}$ and $\frac{3}{7}$, are taken, and it is required to find which is the greater, the question is not readily answered. For although by (131) $\frac{1}{9}$ is less than $\frac{1}{7}$, it is not easy to say how much less; nor is it easy to say whether 5 of the ninths may not be greater than 3 of the sevenths. If both had the same denominator, and if the question were, "How much greater is $\frac{5}{9}$ than $\frac{3}{9}$?" the answer would clearly be $\frac{2}{9}$. But in their present form the values of the two fractions cannot be compared.

200. The same principle applies here as in the Addition and Subtraction of Integers. It is explained in (30) that two concrete numbers cannot be added or subtracted unless they refer to the same magnitude. Thus 5 shillings and 4 pence do not make 9 of either, but if we call the 5 shillings 60 pence, then we may say 60 pence + 4 pence = 64 pence. In this case we could not effect addition until we had found, for the objects to be added, a name which applied to both. In other words, we reduced them to a "common denominator," and then the process became one of Simple Addition.

Now the same thing is needed in adding or subtracting fractions. We want to find a name which will apply equally to each of them. It has been shown (195) that all fractions admit of being expressed in an infinite number of forms, by applying the same multiplier to numerator and denominator. It has also been shown (145) that a common multiple may always be found for any set of numbers. Therefore, if we choose a common multiple of all the denominators this number will serve as the denominator for each of the fractions.

Suppose, for example, we wish to compare together $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$: we observe that 60 is a common multiple of the three denominators; and because 60 is a multiple of 3, any fraction having 3 for its denominator may be so multiplied as to have 60 for its denominator (196). So also because 60 is a multiple of 4 and 5, any fractions whatever, which have 4 or 5 for their denominators, may be expressed with the denominator 60. Now no smaller number than 60 would have served the purpose here, because 60 is the least common multiple of the three denominators. Hence,—

201. *We may take any multiple whatever of a denominator as the denominator of an equivalent fraction, but when two or more fractions are compared we must take a common multiple of all the denominators.*

Example.—Reduce $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{6}{7}$ to a common denominator. We first choose the product of the three denominators to serve as the common denominator of all three. This number is 105.

$$\frac{2}{3} = \frac{2 \times 5 \times 7}{3 \times 5 \times 7} = \frac{70}{105}$$

$$\frac{4}{5} = \frac{4 \times 3 \times 7}{5 \times 3 \times 7} = \frac{84}{105}$$

$$\frac{6}{7} = \frac{6 \times 3 \times 5}{7 \times 3 \times 5} = \frac{90}{105}$$

Now first we want a fraction which shall be equal to $\frac{2}{3}$, and yet which shall have 105 for its denominator. To make 105 the denominator, — 3 has been multiplied by 5 × 7. But unless (194) whatever is done to the denominator is done to the numerator, the fraction will not remain the same. Wherefore we multiply the 2 also by 5 × 7;

$$\text{and } \frac{2}{3} = \frac{2 \times 5 \times 7}{3 \times 5 \times 7} = \frac{70}{105}.$$

Again, in the case of the second fraction, $\frac{4}{5}$, we ask ourselves, "What has been done to the denominator 5 to make it 105?" The answer is, "It has been multiplied by 3 × 7." We infer, therefore, that the numerator also must be multiplied by 3 × 7.
Hence—

$$\frac{4}{5} = \frac{4 \times 3 \times 7}{5 \times 3 \times 7} = \frac{84}{105}.$$

In the third place we propose the same question, "What has been done to the denominator 7 to make it 105?" It has been multiplied by 3 × 5. Therefore the numerator 6 must be multiplied in the same manner, and $\frac{6}{7} = \frac{6 \times 3 \times 5}{7 \times 3 \times 8}$ or $\frac{90}{105}$.

202. When fractions are thus reduced into the same form it is easy to compare them, to add them together, or to find the difference between any two of them. Thus, in the example just given, it appears that $\frac{6}{7}$ is the greatest fraction, for it contains 90 "one hundred and fifths," while $\frac{4}{5}$ contains 84, and $\frac{2}{3}$ contains 70 of them. But until they are thus reduced we cannot in any way compare them accurately.

General Formula.— $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, = \frac{adfh}{bdfh}, \frac{cbfh}{bdfh}, \frac{ebdh}{bdfh}, \frac{gbdf}{bdfh}.$

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR—

RULE.

203. Multiply all the denominators together for the common denominator. Then multiply each numerator by all the denominators except its own, for each of the successive numerators required

Example.—Reduce $\frac{4}{5}$, $\frac{12}{7}$, $\frac{6}{10}$, and $\frac{3}{4}$, to a common denominator.

$$\frac{4}{5} = \frac{4 \times 7 \times 10 \times 4}{5 \times 7 \times 10 \times 4} = \frac{1120}{1400} \quad \frac{6}{10} = \frac{6 \times 5 \times 7 \times 4}{10 \times 5 \times 7 \times 4} = \frac{840}{1400}.$$

$$\frac{12}{7} = \frac{12 \times 5 \times 10 \times 4}{7 \times 5 \times 10 \times 4} = \frac{2400}{1400} \quad \frac{3}{4} = \frac{3 \times 5 \times 7 \times 10}{4 \times 5 \times 7 \times 10} = \frac{1050}{1400}.$$

EXERCISE LIX.

☞ Reduce the following fractions to a common denominator:—

1. $\frac{2}{7}$, $\frac{3}{8}$, and $\frac{4}{5}$.
2. $\frac{7}{12}$, $\frac{3}{16}$, and $\frac{1}{14}$.
3. $\frac{5}{19}$, $\frac{6}{17}$, and $\frac{3}{20}$.
4. $\frac{4}{7}$, $\frac{15}{16}$, $\frac{3}{8}$, and $\frac{4}{17}$.
5. $\frac{7}{12}$, $\frac{3}{13}$, $\frac{6}{16}$, and $\frac{2}{3}$.
6. $\frac{3}{11}$, $\frac{4}{120}$, $\frac{8}{19}$, and $\frac{7}{15}$.
7. $\frac{2}{15}$, $\frac{6}{25}$, and $\frac{4}{5}$.
8. $\frac{5}{16}$, $\frac{4}{19}$, and $\frac{3}{17}$.
9. $\frac{14}{15}$, $\frac{17}{19}$, and $\frac{21}{22}$.
10. $\frac{13}{17}$, $\frac{12}{21}$, $\frac{3}{100}$, and $\frac{6}{19}$.
11. $\frac{7}{15}$, $\frac{8}{20}$, $\frac{3}{13}$, and $\frac{5}{14}$.
12. $\frac{3}{17}$, $\frac{8}{24}$, $\frac{13}{15}$, and $\frac{4}{60}$.

204. The process just described is applicable to all fractions whatever, but is not always the best, for it is desirable (197) in dealing with fractions to express them by numbers as small as possible, as they can in such cases be more readily dealt with. Therefore, whenever the denominators of a fraction have a less common multiple than their product, the least common multiple will be a better common denominator than that which would be obtained by the last rule.

$$\frac{5}{12} = \frac{5 \times 5}{12 \times 5} = \frac{25}{60}$$

$$\frac{3}{15} = \frac{3 \times 4}{15 \times 4} = \frac{12}{60}$$

$$\frac{1}{6} = \frac{1 \times 10}{6 \times 10} = \frac{10}{60}$$

$$\frac{7}{10} = \frac{7 \times 6}{10 \times 6} = \frac{42}{60}$$

For instance, if it be required to reduce $\frac{5}{12}$, $\frac{3}{15}$, $\frac{1}{6}$, and $\frac{7}{10}$, to a common denominator, we may by (169) find that 60 is the least common multiple of all the denominators. Then, in the first case, because 12 is multiplied by 5 to make 60, the numerator is also multiplied by 5; and because in the fraction $\frac{3}{15}$, 15 is multiplied by 4, the numerator 3 must also be multiplied by 4. So also on finding that the denominator of the third fraction has been increased 10 times, we increase the 1, ten times, and the last fraction, $\frac{7}{10}$, has both numerator and denominator increased 6 times.

205. Two things only require to be kept in view in this and the former rule.

I. The common denominator chosen must always be a common multiple of all the denominators.

II. Whatever is done to the denominator of each fraction to produce the common denominator must also be done to the numerator of that fraction.

206.—TO REDUCE FRACTIONS TO A COMMON DENOMINATOR WHEN THEIR DENOMINATORS HAVE A LESS COMMON MULTIPLE THAN THEIR PRODUCT—

RULE.

Find the least common multiple of the denominators for the new denominator, divide this multiple by each of the denominators in succession, and multiply each numerator by the quotient thus found.

Example.—Reduce $\frac{7}{18}$ $\frac{5}{20}$, $\frac{4}{5}$, $\frac{7}{12}$, and $\frac{9}{10}$ to a common denominator.

.By (106) it may be found that 180 is the least common multiple of the denominators 18, 20, 5, 12, and 10.

$\frac{7}{18} = \frac{7 \times 10}{18 \times 10} = \frac{70}{180}$. $\left\{\begin{array}{l}\text{Here the numerator}\\ \text{is multiplied by 10}\end{array}\right\}$ Because $180 \div 18 = 10$.

$\frac{5}{20} = \frac{5 \times 9}{20 \times 9} = \frac{45}{180}$. .... 9 .... $180 \div 20 = 9$.

$\frac{4}{5} = \frac{4 \times 36}{5 \times 36} = \frac{144}{180}$. .... 36 .... $180 \div 5 = 36$.

$\frac{7}{12} = \frac{7 \times 15}{12 \times 15} = \frac{105}{180}$. .... 15 .... $180 \div 12 = 15$.

$\frac{9}{10} = \frac{9 \times 18}{10 \times 18} = \frac{162}{180}$. .... 18 .... $180 \div 10 = 18$.

EXERCISE LX.

☞ Reduce the following fractions to a common denominator:—

1. $\frac{3}{15}$ and $\frac{4}{10}$.
2. $\frac{7}{12}$ and $\frac{4}{18}$.
3. $\frac{6}{20}$, $\frac{3}{15}$, and $\frac{4}{7}$.
4. $\frac{5}{9}$, $\frac{3}{21}$, and $\frac{7}{12}$.
5. $\frac{4}{18}$, $\frac{6}{24}$, and $\frac{4}{5}$.
6. $\frac{4}{24}$, $\frac{3}{10}$, and $\frac{15}{60}$.
7. $\frac{2}{15}$, $\frac{3}{75}$, and $\frac{19}{21}$.
8. $\frac{2}{11}$, $\frac{3}{15}$, and $\frac{14}{44}$.
9. $\frac{2}{7}$, $\frac{3}{18}$, $\frac{14}{16}$, and $\frac{12}{28}$.
10. $\frac{2}{12}$, $\frac{14}{24}$, $\frac{7}{15}$, and $\frac{3}{20}$.
11. $\frac{6}{100}$, $\frac{4}{24}$, $\frac{7}{18}$, and $\frac{19}{150}$.
12. $\frac{3}{7}$, $\frac{4}{14}$, and $\frac{12}{2100}$.
13. $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, and $\frac{5}{11}$.
14. $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{12}$.
15. $\frac{1}{7}$, $\frac{1}{8}$, and $\frac{1}{9}$.

SECTION II.—ADDITION AND SUBTRACTION.

207. ***Whenever fractions have the same denominator they may be added together by adding their numerators only.***

Demonstrative Example.—$\frac{4}{9} + \frac{6}{9} + \frac{1}{9} = \frac{4+6+1}{9} + \frac{11}{9}$

The denominator is the *namer* of the parts into which the unit is divided; consequently all the fractions which have the same denominator refer to the same parts of unity.

General Formula.—$\frac{a}{b} + \frac{c}{b} + \frac{x}{b} = \frac{a+c+x}{b}.$

208. *Observation.*—Sometimes the numbers to be added together are improper fractions or mixed numbers. If the former, they should be reduced to mixed numbers, and the whole numbers should then be added by themselves, and also the fractions by themselves.

Example.—Find the sum of $5\frac{3}{4}$, $8\frac{2}{10}$, $12\frac{4}{5}$, $2\frac{3}{8}$.

By (9) these may be added in any order, thus:—

$5\frac{3}{4} + 8\frac{2}{10} + 12\frac{4}{5} + 2\frac{3}{8} = 5 + 8 + 12 + 2 + \frac{3}{4} + \frac{2}{10} + \frac{4}{5} + \frac{3}{8}.$

Adding together the whole numbers and reducing the fractions to a common denominator, we have—

$$27 + \frac{30}{40} + \frac{8}{40} + \frac{32}{40} + \frac{15}{40} = 27\frac{85}{40} = 29\frac{5}{40}.$$

RULE FOR ADDITION OF FRACTIONS.

209. Reduce the fractions to a common denominator by (203); add the numerators only, and place the common denominator under the sum.

EXERCISE LXI.

1. $\frac{3}{5} + \frac{4}{7}$.
2. $\frac{3}{12} + \frac{5}{16} + \frac{4}{17}$.
3. $\frac{2}{7} + \frac{14}{15} + \frac{1}{3}$.
4. $2\frac{7}{12} + 3\frac{4}{5}$.
5. $17\frac{1}{3} + 6\frac{1}{9} + \frac{4}{13}$.
6. $\frac{2}{10} + \frac{12}{55} + \frac{103}{112}$.
7. $4\frac{1}{5} + 6\frac{2}{7} + \frac{13}{25}$.
8. $4\frac{1}{4} + \frac{16}{5}$.
9. $\frac{13}{4} + \frac{17}{5} + \frac{1}{3}$.
10. $\frac{11}{10} + \frac{5}{100} + \frac{39}{100000}$.
11. $\frac{4}{15} + \frac{6}{17} + \frac{3}{11}$.
12. $\frac{17}{12} + 2\frac{1}{5} + \frac{15}{9}$.
13. $\frac{284}{5} + \frac{61}{12} + \frac{6}{7}$.
14. $\frac{13}{6} + 14\frac{5}{8} + \frac{21}{23}$.
15. $\frac{16}{15} + \frac{4}{7} + 100\frac{3}{5}$.
16. $\frac{4}{10} + 10\frac{1}{4} + 3\frac{1}{5}$.
17. $21\frac{2}{14} + \frac{2}{3} + 1\frac{5}{81}$.
18. $12\frac{1}{6} + \frac{21}{4} + \frac{3}{15}$.

19. How much of a ship does a person own who has purchased at different times $\frac{1}{3}$, $\frac{3}{8}$, $\frac{1}{4}$, and $\frac{7}{40}$ of it?

20. Add together $2519\frac{5}{23}$, $184\frac{3}{11}$, and $2091\frac{7}{18}$.

SUBTRACTION OF VULGAR FRACTIONS.

210. ***Whenever two fractions have the same denominator, their difference can be found by finding the difference between their numerators.***

Demonstrative Example.—$\frac{12}{17} - \frac{8}{17} = \frac{12-8}{17} = \frac{4}{17}$

General Formula.—$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$

211. *Observation.*—If the numbers be mixed numbers it is more convenient first to reduce them to improper fractions, and then perform the subtraction as in the rule.

Example.—$7\frac{2}{3} - 3\frac{4}{5}$. Here it would not be easy, as in Addition, to deal with the whole numbers by themselves, because $\frac{4}{5}$ cannot be taken from $\frac{2}{3}$.

By (185) $7\frac{2}{3} = \frac{23}{3}$, and $3\frac{4}{5} = \frac{19}{5}$.
By (203) $\frac{23}{3} = \frac{115}{15}$, and $\frac{19}{5} = \frac{57}{15}$.
By (210) $\frac{115}{15} - \frac{57}{15} = \frac{115-57}{15} = \frac{58}{15}$.
By (187) $\frac{58}{15} = 3\frac{1}{15}$ = the answer.

RULE.

Reduce them to a common denominator (203); subtract the less numerator from the greater, and place the common denominator underneath this difference.

EXERCISE LXII.

☞ Find the difference between—

1. $\frac{7}{15}$ and $\frac{3}{12}$.
2. $\frac{24}{35}$ and $\frac{32}{24}$.
3. $\frac{17}{19}$ and $\frac{5}{7}$.
4. $\frac{6}{18}$ and $\frac{3}{4}$.
5. $\frac{7}{10}$ and $\frac{13}{15}$.
6. $\frac{2}{19}$ and $1\frac{7}{12}$.
7. $21\frac{3}{5} - 17\frac{7}{9}$
8. $1\frac{4}{7} - \frac{17}{58}$.
9. $\frac{27}{4} - \frac{3}{16}$.
10. $14\frac{1}{7} - 3\frac{2}{5}$.
11. $26\frac{2}{5} - 18\frac{12}{13}$.
12. $\frac{103}{12} - \frac{12}{103}$.
13. $172\frac{1}{8} - \frac{97}{4}$.
14. $21\frac{2}{15} + 7\frac{1}{9}$.
15. $\frac{43}{7} + \frac{3}{12} - \frac{5}{16}$.
16. Find the sum and difference of $\frac{2}{3}$ and $\frac{5}{6}$.
17. To what number can I add $7\frac{3}{5}$ so as to make $24\frac{7}{12}$?
18. By how much does the sum of $30\frac{5}{8}$ and $6\frac{2}{7}$ exceed the sum of $10\frac{3}{11}$ and $\frac{1792}{18}$?
19. Which is the greater, and by how much, $(\frac{1}{7} + \frac{2}{19})$ or $(\frac{5}{32} - \frac{3}{22})$?
20. A person had £$1056\frac{2}{3}$ of a certain stock; he sold out successively £$84\frac{1}{4}$, £$56\frac{3}{16}$, and £$195\frac{3}{8}$, and bought in £$278\frac{3}{4}$: how much did he at last possess?

21. By how much does $\frac{8}{7}$ exceed $\frac{7}{8}$?

22. How much less than £500 does a person lay out who purchases one share in each of the following railways:—London and North-Western at £$100\frac{3}{4}$, Great Western at £$64\frac{7}{8}$, Great Northern at £$86\frac{1}{4}$, Eastern Counties at £$9\frac{7}{16}$, Midland at £$74\frac{3}{8}$?

23. How much is lost on each £50 share in a Mining Company when the price is £$36\frac{5}{16}$?

24. A gentleman presents to his daughter on her wedding day the following property:—$\frac{3}{5}$ of a ship which was valued at £13,000, $\frac{3}{8}$ of a factory worth £42,000, and a cheque to make up the whole £20,000; what was the amount of the cheque?

25. How much is required to make the difference between $2\frac{3}{7}$ and $1\frac{3}{42}$, equal to the sum of $\frac{1}{14}$, $1\frac{5}{9}$, and $\frac{3}{81}$?

SECTION III.—MULTIPLICATION OF VULGAR FRACTIONS.

212. Multiplication has been described (56) as a method of repeating a number, or taking it a certain number of times. In all cases, therefore, in which *integer* numbers are the multipliers, the effect of multiplication is to increase the multiplicand.

But a fraction only represents certain parts of a unit; while, therefore, to multiply by an integer is to take the multiplicand a certain number of times, to multiply by a fraction is to take the multiplicand certain *parts* of a time. Hence the effect of multiplying by a number less than unity is to diminish the multiplicand. The definition given of Multiplication in (57) requires to be somewhat expanded in order to meet this case.

213. To multiply one number by another is—

I. To repeat the multiplicand as many times, *or parts of a time*, as there are units in the multiplier.

II. To find a number which is as many times more *or less* than the multiplicand as the multiplier is more *or less* than unity.

III. To do to the multiplicand whatever has been done to unity to make the multiplier.

From this it appears that to multiply by a fraction, say by $\frac{3}{4}$, is to take the given number $\frac{3}{4}$ of a time, or to take $\frac{3}{4}$ of it. The word *of*, placed between two fractions, means exactly the same as the sign (×) of Multiplication.

$$\text{Thus, } \frac{5}{9} \text{ of } \frac{11}{12} = \frac{5}{9} \times \frac{11}{12}.$$

214. *We multiply by a fraction when we multiply by its numerator and divide by its denominator.*

215. *Demonstrative Example I.*—Let it be required to multiply 10 by $\frac{6}{7}$ or to take $\frac{6}{7}$ of 10. We first take $\frac{1}{7}$ of 10, or divide 10 by 7; by (192) this is $\frac{10}{7}$; but it was required to take *six* sevenths of it, therefore we multiply this by 6, and $\frac{10}{7} \times 6 = \frac{10 \times 6}{7} = \frac{60}{7}$.

216. *Demonstrative Example II.*—Let it be required to multiply $\frac{4}{5}$ by $\frac{7}{9}$, that is to say (182) to multiply $\frac{4}{5}$ by the ninth part of 7. We first multiply it by 7. Now to multiply a fraction by a whole number is (191) to multiply its numerator. Wherefore, $\frac{4}{5}$ multiplied by 7 equals $\frac{28}{5}$. But it was not required to multiply by 7, but by the ninth part of 7. Wherefore $\frac{28}{5}$ is nine times too great, and the required answer must be one-ninth of this fraction. But (192) to take one-ninth of a fraction is to multiply its denominator by 9, and $\frac{1}{9}$ of $\frac{28}{5} = \frac{28}{9 \times 5}$ or $\frac{28}{45}$. But this answer would have been obtained at once by multiplying by the numerator and dividing by the denominator.

217. *Demonstrative Example III.*—Suppose it be required to multiply $\frac{7}{12}$ by $\frac{3}{8}$. This is equivalent to taking $\frac{3}{8}$ of $\frac{7}{12}$ (213). Let us first take $\frac{1}{8}$ of $\frac{7}{12}$, that is to say, divide $\frac{7}{12}$ by 8. Now by (192), to divide a fraction is to multiply its denominator, whence $\frac{7}{12} \div 8 = \frac{7}{12 \times 8}$ or $\frac{7}{96}$. One eighth of the fraction $\frac{7}{12}$ has now been taken. But it was not required to find $\frac{1}{8}$, but $\frac{3}{8}$; wherefore $\frac{7}{96}$ is 3 times too little, and we must multiply it by 3. But (191) to multiply a fraction by 3 we must multiply its numerator, and $\frac{7}{96} \times 3 = \frac{7 \times 3}{96} = \frac{21}{96}$. But this is the answer which would have been found at once by multiplying the numerators together and also the denominators.

218. *Demonstrative Example IV.*—To multiply $\frac{4}{9}$ by $\frac{8}{11}$, we have (213) to do to the multiplicand ($\frac{4}{9}$) whatever has been done to 1 in order to make $\frac{8}{11}$. But to make $\frac{8}{11}$, unity has been multiplied by 8 and divided by 11. We have therefore to multiply $\frac{4}{9}$ by 8 and divide it by 11. Wherefore $\frac{4}{9} \times \frac{8}{11} = \frac{8 \times 4}{9 \times 11} = \frac{32}{99}$.

General Formula.—$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$; or, $\frac{a}{b}$ of $\frac{c}{d} = \frac{ac}{bd}$.

It generally saves trouble to bring all mixed numbers into improper fractions before multiplying them, and afterwards to reduce the answer, if an improper fraction, to a mixed number.

RULE.

219. Multiply the numerators together for a new numerator, and the denominators for a new denominator.

220. *Observation I.*—Two inferences can easily be deduced from the explanation in (212). I. That when a number is multiplied by a proper fraction, the answer is always as much less than the multiplicand as the numerator of the multiplier is less than its denominator; and II., that the product of any two or more proper fractions is less than either of the factors.

Observation II.—Whenever the same numbers occur in the numerators and denominators of the fractions which are to be multiplied or compounded, they may be cancelled or struck out, by (136).

EXERCISE LXIII.

☞ Simplify the following expressions :—

1. $\frac{4}{7} \times \frac{2}{5}$; $\frac{8}{15} \times \frac{12}{17}$; $\frac{5}{9} \times \frac{20}{7} \times 2\frac{1}{4}$.
2. $\frac{6}{15} \times \frac{3}{5} \times \frac{14}{7}$; $\frac{21}{8} \times \frac{14}{6}$; $3\frac{2}{5} \times 5\frac{4}{9}$.
3. $\frac{1}{7}$ of $\frac{3}{8}$ of $\frac{5}{11}$; $\frac{2}{9}$ of $\frac{5}{16}$; $\frac{4}{7} \times \frac{2}{3}$ of $\frac{3}{4}$.
4. $2\frac{1}{4} \times \frac{3}{7}$ of $\frac{4}{12}$; $\frac{4}{7} \times 3\frac{1}{2} \times \frac{15}{4}$; $41\frac{7}{10} \times \frac{2}{7} \times \frac{5}{9}$.
5. $(\frac{3}{7} + \frac{5}{6}) \times (\frac{2}{5}$ of $\frac{4}{9})$; $\frac{4}{17}$ of $\frac{2}{9} - \frac{1}{19}$ of $\frac{1}{10}$.
6. $(4\frac{1}{3} + 2\frac{1}{7}) - (1\frac{2}{5} \times \frac{1}{3})$; $(\frac{5}{7}$ of $\frac{1}{9}) + (\frac{2}{5} \times \frac{4}{17})$.
7. $100\frac{3}{5} \times 253\frac{1}{7}$; $(409\frac{3}{8} + 2\frac{2}{9}) \times 15\frac{6}{7}$; $31\frac{4}{17} \times \frac{10}{23}$.
8. $\frac{2}{7}$ of $\frac{3}{4}$ of 100; $(\frac{5}{16}$ of $12) \times (\frac{3}{14}$ of $7)$; $\frac{2}{5} \times \frac{7}{8}$ of $\frac{9}{10}$.
9. What is the product of the sum and difference of $\frac{2}{11}$ and $\frac{4}{15}$?
10. How much must be added to $\frac{2}{17}$ of 50 to make $\frac{1}{12}$ of $\frac{3}{5}$ of 1250?
11. From what number can I subtract the product of $\frac{16}{18}$ and $\frac{8}{14}$ so that the remainder may be the sum of $\frac{1}{13}$ and $\frac{51}{60}$?
12. How much is required to buy 8 shares in a canal at £$45\frac{3}{16}$ each?
13. What does a person lose who buys seventeen £100 shares at $\frac{5}{16}$ premium, and sells them at £$86\frac{2}{5}$ each; and what does another gain who sells at the same price the same number of shares bought at $79\frac{7}{16}$ each?
14. What is the difference between the product of $2\frac{1}{3}$ and $3\frac{15}{19}$, and that of $\frac{8}{15}$ and $19\frac{5}{8}$?
15. What is the difference between the sum and product of $\frac{1}{3}$ and $\frac{2}{19}$?
16. From a field of $12\frac{3}{5}$ acres are taken off 3 equal portions of $1\frac{7}{40}$ acres each, and also two portions of 2 roods 15 poles each; what part of the field remains?
17. How much has a person left out of £20 who purchases 19 United States dollars, worth each $\frac{5}{24}$ of £1 sterling; 32 Prussian thalers, worth $\frac{28}{200}$ of £1, and 18 Danish rix-dollars, each worth $\frac{11}{100}$ of £1?

SECTION IV.—DIVISION OF VULGAR FRACTIONS.

221. The word Division throughout Integral Arithmetic conveys the notion of diminution; for (89) the answer to a Division sum is always as much less than the dividend as the divisor is greater than one. As the divisor is diminished the quotient is increased; therefore, if the divisor be less than unity the quotient will be greater than the dividend; and this is the case whenever a proper fraction is the divisor. Hence, dividing by a fraction increases the dividend just as multiplying by a fraction diminishes the multiplicand. The definitions of Division given in (88) need only be very slightly extended to meet this case.

222. To divide one number by another is—

I. To find how many times *or parts of a time* the divisor is contained in the dividend.

II. To find a number which is as many times more *or less* than the dividend as unity is more *or less* than the divisor.

III. To find a multiplier which, if applied to the divisor, would produce the dividend.

223. Before proceeding to the examination of the Rule it is necessary to understand the meaning of the term RECIPROCAL. 1 multiplied by 20 is the reciprocal of 1 divided by 20 ($\frac{1}{20}$), and *vice versâ*. So if 1 be multiplied by 7 and divided by 9, the fraction $\frac{7}{9}$ is the result; but if, instead of this, 1 were multiplied by 9 and divided by 7, the fraction $\frac{9}{7}$ would result. Now $\frac{7}{9}$ and $\frac{9}{7}$ are each the reciprocal of the other, and by inverting the terms of a fraction we always obtain its reciprocal.

Thus $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$; 16 of $\frac{1}{16}$; $\frac{11}{100}$ of $\frac{100}{11}$.

224. ***If the relation of one magnitude to another is expressed by a number, the relation of the second to the first is expressed by the reciprocal of that number.***

Demonstrative Example.—Because a florin is $\frac{4}{5}$ of half a crown, half a crown is $\frac{5}{4}$ of a florin; and because $12 = \frac{3}{4}$ of 16, $16 = \frac{4}{3}$ of 12.

General Formula.—If x be $\frac{m}{n}$ of y, then y is $\frac{n}{m}$ of x.

EXERCISE LXIV.

☞ What are the reciprocals of the following numbers?—

1. 7; $\frac{1}{9}$; $\frac{14}{15}$. 2. $2\frac{3}{8}$; $\frac{17}{8}$; $15\frac{4}{11}$. 3. $21\frac{1}{3}$; $\frac{6}{7}$; $108\frac{1}{8}$.

225. *We divide by a fraction when we multiply by its reciprocal.*

Demonstrative Example I.—Divide 8 by $\frac{1}{7}$; *i. e.*, find how many times $\frac{1}{7}$ is contained in 8. But because $\frac{7}{7}$ make one unit, $\frac{1}{7}$ is contained 7 times in 1, and is therefore contained 7 × 8 or 56 times in 8. Hence the answer is 56. Dividing 8 by $\frac{1}{7}$ is the same as multiplying 8 by 7, and 7 is the reciprocal of $\frac{1}{7}$.

226. *Demonstrative Example II.*—Divide 12 by $\frac{3}{5}$; *i.e.*, find how many times $\frac{3}{5}$ are contained in 12. Now because $\frac{1}{5}$ is contained 5 times in 1, it is contained 60 times in 12; therefore three-fifths are contained one-third of 60 times in 12. Hence $\frac{12 \times 5}{3} = 20 =$ the number of times $\frac{3}{5}$ are contained in 12; or $12 \div \frac{3}{5} = 12 \times \frac{5}{3}$.

227. *Demonstrative Example III.*—Let it be required to divide $\frac{3}{4}$ by $\frac{12}{15}$; or to divide $\frac{3}{4}$ by the fifteenth part of 12. First divide it by 12. Now (192) to divide by 12 is to multiply the denominator by 12. Hence $\frac{3}{4} \div 1 = \frac{3}{4 \times 12} = \frac{3}{48}$. But it was not required to divide by 12, but by the fifteenth part of 12; therefore in dividing by 12 we have made it 15 times too little; $\frac{3}{48}$ must therefore be multiplied by 15 to rectify this error. But by (191) $\frac{3}{48} \times 15 = \frac{3 \times 15}{48} = \frac{45}{48}$. But this fraction equals $\frac{3}{4}$ multiplied by $\frac{15}{12}$, or by the reciprocal of $\frac{12}{15}$.

228. *Demonstrative Example IV.*—To divide $\frac{2}{13}$ by $\frac{3}{20}$ is (222) to find how often the latter is contained in the former. To do this we may bring them (203) to a common denominator, then the sum stands—Divide $\frac{40}{260}$ by $\frac{39}{260}$. But to find how many times $\frac{40}{260}$ contains $\frac{39}{260}$ is the same thing as to find how many times 40 oranges contain 39 oranges, or how many times £40 contain £39, or how many times 40 contains 39. Now (182) the fraction $\frac{40}{39}$ represents the number of times that 40 contains 39. Wherefore $\frac{2}{13} \div \frac{3}{20} = \frac{40}{39}$, or $\frac{2}{13} \times \frac{20}{3}$.

229. *Demonstrative Example V.*—It is required to divide $\frac{8}{9}$ by $\frac{3}{11}$. Now (222) this is to find a number which, if multiplied by $\frac{3}{11}$, will give $\frac{8}{9}$ as product. Therefore (213) $\frac{3}{11}$ of this unknown fraction must equal $\frac{8}{9}$.

But by (224) if $\frac{8}{9} = \frac{3}{11}$ of the required fraction, that fraction must equal $\frac{11}{3}$ of $\frac{8}{9}$. And $\frac{11}{3}$ of $\frac{8}{9} = \frac{11}{3} \times \frac{8}{9}$, or $\frac{88}{27}$.

Therefore $\frac{8}{9} \div \frac{3}{11} = \frac{8}{9} \times \frac{11}{3}$.

General Formula.—$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

RULE TO DIVIDE BY A FRACTION.

230. Find the reciprocal of the divisor (Ex. LXIV.) and then multiply the dividend by it.

Observation.—Sometimes a question in Division takes this form :—Resolve $\frac{\frac{27}{5}}{\frac{8}{9}}$ into a simple fraction. Fractions in this shape are often called Complex Fractions. But as it is evidently intended that the upper fraction should be divided by the lower, the problem $\frac{\frac{27}{5}}{\frac{8}{9}}$ is the same as $\frac{27}{5} \div \frac{8}{9}$, or $\frac{27}{5} \times \frac{9}{8}$; and no new rule or explanation is needed in this ase. Such a complex fraction would be read, $\frac{27}{5}$ *upon* or *by* $\frac{8}{9}$.

EXERCISE LXV.

☞ Solve the following expressions :—

1. $\frac{2}{7} \div \frac{5}{9}$; $\frac{4}{10} \div \frac{3}{15}$; $\frac{12}{27} \div \frac{6}{19}$.
2. $2\frac{1}{7} \div 1\frac{2}{5}$; $41\frac{1}{10} \div \frac{15}{16}$; $\frac{27}{5} \div \frac{14}{15}$.
3. $\frac{\frac{2}{7}}{\frac{4}{9}}$; $\frac{\frac{2}{3} \text{ of } \frac{5}{6}}{\frac{1}{3} + \frac{1}{9}}$; $\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$; $(3\frac{1}{2} + 18\frac{1}{5}) \div (\frac{2}{7} \text{ of } \frac{10}{21})$.
4. $\frac{14\frac{1}{3} + \frac{1}{6}}{2\frac{1}{3} + \frac{5}{9}}$; $\frac{2\frac{1}{4} \times 3\frac{1}{7}}{\frac{3}{7} \text{ of } \frac{2}{9}}$; $\frac{26\frac{3}{7} - 1\frac{13}{21}}{\frac{4}{9} + \frac{6}{5} - \frac{1}{3}}$.
5. $28\frac{1}{3} \div 16\frac{4}{5}$; $[108\frac{3}{12} + (\frac{4}{7} \text{ of } \frac{3}{8}) - 2\frac{1}{9}] - \frac{5}{18} \times \frac{3}{4}$.
6. Divide the product of $\frac{2}{5}$ and $\frac{4}{15}$ by their sum.
7. What number multiplied by $\frac{7}{8}$ will give $2\frac{3}{7}$?
8. What fraction divided by $3\frac{5}{13}$ will give $7\frac{1}{4}$?
9. Divide the product of $\frac{7}{15}$ and $2\frac{2}{3}$ by the reciprocal of their sum.
10. To the sum of $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{10}$, add the reciprocal of $15\frac{1}{15}$.
11. From 18 take its fourth, its seventh, and its eleventh, and divide he remainder by the product of $4\frac{7}{8}$ and $2\frac{1}{5}$.
12. To $\frac{7}{12}$ of 30 add $\frac{5}{13}$ of $80\frac{1}{2}$, and divide the result by $4\frac{7}{8}$.
13. Give the sum, difference, product, and quotient of $\frac{5}{8}$ and $\frac{2}{3}$, the latter fraction being the divisor.
14. Divide $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{5}{7}$ of 42 by the sum of $2\frac{1}{5}$ and $4\frac{3}{5}$.
15. A gentleman inherited $\frac{2}{5}$ of a certain mercantile concern, of which he sold one-half, and the remainder he left on his death to his three daughters in equal portions : what had each daughter, seeing that the whole concern was valued at £154,000?

SECTION V.—REDUCTION OF VULGAR FRACTIONS TO OTHERS OF DIFFERENT DENOMINATIONS.

231. It is often useful to represent the same quantity fractionally in a variety of ways, thus: to find what fraction of a pound $\frac{5}{7}$ of a shilling is; or to find what part of a shilling is equal to $\frac{5}{7}$ of a pound. The rule for solving such questions depends on the following considerations.

232. *If one quantity be a certain fractional part of another, it is a greater fractional part of that which is less than that other, and a smaller fractional part of whatever is greater than that other.*

Demonstrative Example.—$\frac{2}{7}$ of £1 is more than $\frac{2}{7}$ of a shilling, but it is less than $\frac{2}{7}$ of £5.

A sum of money which is a certain fraction of a pound is a greater fraction of a shilling, in just the same degree as the shilling is less than a pound. That is to say, $\frac{2}{7}$ of £1 is $\frac{2}{7}$ of 20s., and is therefore 20 times $\frac{2}{7}$ of one shilling, or $\frac{20 \times 2}{7}$, or $\frac{40}{7}$ of 1s. So also if a certain sum of money is $\frac{4}{5}$ of a shilling, it is a less fraction of £1 in just the same degree as a shilling is less than a pound; that is, because $\frac{4}{5}$ of a shilling is $\frac{4}{5}$ of $\frac{1}{20}$ of a pound; $\frac{4}{5}$ of a shilling $= \frac{4}{5 \times 20}$ or $\frac{4}{100}$ of £1.

General Formula.—If a be $\frac{n}{m}$ of b it is $\frac{pn}{m}$ of $\frac{b}{p}$, and it is $\frac{n}{pm}$ of pb.

RULE.

233. When the fraction is to be altered to one of a higher name, multiply the denominator by as many of the less as make one of the greater.

But when the fraction is to be altered to an equivalent fraction of a lower name, multiply the numerator by as many of the lower name as make one of the higher.

Example I.—What fraction of a yard is $\frac{3}{8}$ of an inch?

Here because $\frac{3}{8}$ of an inch has to be expressed as a fraction of a length greater than an inch, the new fraction must be as many times less than $\frac{3}{5}$ as an inch is less than a yard; or $\frac{3}{5}$ of an inch $= \frac{3}{5}$ of $\frac{1}{36} = \frac{3}{5 \times 36}$ $= \frac{3}{180}$ of a yard.

Example II.—What fraction of an hour is $\frac{2}{11}$ of a day?

Here because a day is 24 times greater than an hour any fraction of a day is 24 times that fraction of an hour. Hence $\frac{2}{11}$ of a day $= \frac{2}{11}$ of 24 hours $= \frac{2 \times 24}{11}$ or $\frac{48}{11}$ of an hour.

EXERCISE LXVI.

1. Reduce $\frac{4}{7}$ of a day to the fraction of a year, and of a lunar month.

2. Reduce $\frac{5}{8}$ of a furlong to the fraction of a league, a mile, a pole, a yard, and a foot.

3. How would $\frac{4}{7}$ of a florin be expressed as a fraction of each of the English coins?

4. Reduce $\frac{5}{18}$ of an acre to the fraction of a square mile, and also of a square yard.

5. Reduce $\frac{14}{17}$ of an avoirdupois ounce to the fraction of a cwt., a pound, a troy pound, a troy ounce, and a grain.

6. What portion of 7 yards is $\frac{5}{11}$ of an inch?

7. What fraction of 9 inches is $\frac{5}{78}$ of a mile?

8. If a certain sum of money be $\frac{41}{56}$ of 10s., what fraction is it of 15s., of £1, and of a five-pound note?

9. Express $\frac{8}{15}$ of an hour in terms of a minute, a day, and a week.

10. Express $\frac{18}{25}$ of a gallon in terms of a pint, a puncheon, and a pipe.

11. What part of 100 quarters of wheat is $\frac{5}{7}$ of 9 bushels?

12. What part of 17 square miles is $\frac{2}{3}$ of $\frac{5}{6}$ of 12 acres?

13. Express 2 roods 15 perches as the fraction of an acre, and of a square mile.

14. What part of £50 is $\frac{2}{3}$ of $\frac{3}{10}$ of two shillings?

15. Express $\frac{14}{25}$ of a minute in terms of a second and of a degree.

16. Express the height of Chimilari, which is 29,000 feet, as a fraction of the diameter of the earth, which is 7,926 miles.

17. What fraction of 5 miles is $\frac{3}{8}$ of $\frac{5}{7}$ of 9 yards?

18. The sidereal revolution of the moon is made in 27 days 7 hours 43 minutes and 11 seconds; what fraction is it of the sidereal revolution of the earth, which occupies 365 days 6 hours 9 minutes and 11 seconds?

SECTION VI.—CONTINUED FRACTIONS.

234. When a vulgar fraction is expressed by two large numbers which are prime to each other, it cannot (197) be reduced to a lower name, and yet it is not easy to acquire an exact idea of its value. The process of resolving such an expression into the form of a continued or converging fraction is designed to give a series of fractions expressed in smaller numbers, but approximating as nearly as possible to the value of the given fraction.

A continued fraction has unity for its numerator, and for a denominator a whole number plus *a fraction, which has itself unity for a numerator, and for a denominator a whole number* plus *another similar fraction, and so on.*

The terms of the fraction $\frac{135}{292}$, for example, are irreducible. We may by (194) divide both by 135. Hence* $\frac{135}{292} = \frac{1}{(\frac{292}{135})}$. Here 1 is the numerator, and the denominator is an improper fraction, which by (185) may itself be reduced to a mixed number. Thus we have $\frac{135}{292} = \frac{1}{2\frac{22}{135}}$. Suppose now we neglect the fraction $\frac{22}{135}$, the remaining fraction $\frac{1}{2}$ is a rough approximation to $\frac{135}{292}$, but it is too great, because the whole of the denominator has not been taken into account. But if for $\frac{22}{135}$ we substitute 1, the fraction becomes $\frac{1}{2+1}$ or $\frac{1}{3}$; this is evidently less than the required fraction, which therefore lies between $\frac{1}{2}$ and $\frac{1}{3}$.

In order to obtain a closer approximation we treat the last fraction in the same manner as the former. Thus $\frac{22}{135} = \frac{1}{(\frac{135}{22})} = \frac{1}{6+\frac{3}{22}}$ and the proposed fraction $\frac{135}{292} = \cfrac{1}{2+\cfrac{1}{6+\frac{3}{22}}}$. If we now neglect the $\frac{3}{22}$ we observe that the remaining fraction $\frac{1}{2+\frac{1}{6}}$, or $\frac{1}{(\frac{13}{6})}$, or $\frac{6}{13}$ is too little, for the denominator is too great. Hence we infer that the value of the original fraction lies between $\frac{1}{2}$ and $\frac{6}{13}$. But as (210) the difference between them is $\frac{1}{26}$, the fraction $\frac{6}{13}$ does not differ from the truth by so much as $\frac{1}{26}$.

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* This transformation may be justified also on the general principle explained in (223) concerning the reciprocals of numbers. *Every number whether integral or fractional equals unity divided by its reciprocal; e.g.*, $6 = \frac{1}{\frac{1}{6}}$; $\frac{4}{7} = \frac{1}{\frac{7}{4}}$

If we now deal with the fraction $\frac{3}{22}$ as with the rest, we have this result, $\frac{135}{292} = \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{(\frac{22}{3})}}} = \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{7 + \cfrac{1}{3}}}}$. If now we neglect the $\frac{1}{3}$ and consider the rest of the expression only, we have $6\frac{1}{7} = \frac{43}{7}$, therefore $\frac{1}{6\frac{1}{7}} = \frac{1}{(\frac{43}{7})}$ or $\frac{7}{43}$ and $2 + \frac{1}{6\frac{1}{7}} = 2 + \frac{7}{43}$ or $\frac{93}{43}$. Hence $\cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{7}}} = \frac{1}{(\frac{93}{43})} = \frac{43}{93}$. Now the last expression $\frac{6}{13}$ was less than $\frac{135}{292}$ because it made the denominator appear greater than it really was. But since $\frac{1}{7}$ is greater than $\frac{1}{7 + \frac{1}{3}}$, therefore $\cfrac{1}{6 + \cfrac{1}{7}}$ is less than $\cfrac{1}{6 + \cfrac{1}{7 + \cfrac{1}{3}}}$, and is consequently greater than $\cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{7 + \cfrac{1}{3}}}}$. The new fraction $\frac{43}{93}$ is therefore too great, and the real value of $\frac{135}{292}$ lies between $\frac{6}{13}$ and $\frac{43}{93}$, but as these differ only by $\frac{1}{1209}$ the error committed in calling the original fraction $\frac{43}{93}$ is less than $\frac{1}{1209}$.

The fraction $\frac{135}{292}$ has therefore been resolved into this form, and if we begin at the upper fraction, and take the several denominators successively into account, we have the series of fractions, $\frac{1}{2}, \frac{6}{13}, \frac{43}{93}, \frac{135}{292}$, of which the second is nearer the truth than the first, and the third nearer than the second ; but each is nearer than any other expression formed of numbers equally small.

$$\cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{7 + \cfrac{1}{3}}}}$$

In these Continued Fractions (Chain Fractions, as the Germans call them) it will be seen that the approximation obtained is alternately greater and less than the value of the original quantity. Thus :—

First operation ;	$\frac{1}{2}$ is too great.
Second operation ;	$\frac{6}{13}$ is too small.
Third operation ;	$\frac{43}{93}$ is too great.

It is manifest that the series of denominators, 2, 6, 7, and 3, might have been at once obtained by employing the method (155) for finding the greatest common measure of two numbers.

```
135)292(2
    270
    ───
     22)135(6
        132
        ───
          3)22(7
            21
            ──
             1) 3 (3
```

235. The four converging fractions are thus formed:—

$$\begin{array}{llcll}
\text{First} & = & \dfrac{1}{\text{First quotient}} & = & \dfrac{1}{2} \\[2ex]
\text{Second} & = & \dfrac{1 \times 6}{(2 \times 6) + 1} & = & \dfrac{6}{13} \\[2ex]
\text{Third} & = & \dfrac{(6 \times 7) + 1}{(13 \times 7) + 2} & = & \dfrac{43}{93} \\[2ex]
\text{Fourth} & = & \dfrac{(43 \times 3) \times 6}{(93 + 3) \times 13} & = & \dfrac{135}{292}.
\end{array}$$

The first fraction has its numerator and denominator multiplied by the second quotient, and one added to the denominator in order to make the second fraction. This second fraction has its numerator and denominator multiplied by the third quotient : to these results the numerator and denominator of the first fraction are added, and thus the third fraction is found. The other fractions are obtained by the same method.

TO RESOLVE A FRACTION INTO A CONVERGENT SERIES—

RULE.

236. Transform the fraction into 1 divided by its reciprocal. Resolve this reciprocal into a mixed number, and transform the fractional part of this number into unity divided by its reciprocal. Proceed in this way until 1 is the numerator of the last fraction.

Or, with the numerator and denominator of the fraction proceed as in the rule for finding the greatest common measure; the series of quotients will be the series of denominators required.

Example.—Reduce $\frac{363}{149}$ to the form of a continued fraction.

$$\frac{363}{149} = 2\tfrac{65}{149} \qquad = 2 + \cfrac{1}{(\tfrac{149}{65})} \qquad = 2 + \cfrac{1}{2\tfrac{19}{65}} \qquad \text{1st approx. } \frac{5}{2}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{(\tfrac{65}{19})}} \qquad = 2 + \cfrac{1}{2 + \cfrac{1}{3 + (\tfrac{8}{19})}} \qquad \text{2nd approx. } \frac{17}{7}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{(\tfrac{19}{8})}}} \qquad = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + (\tfrac{3}{8})}}} \qquad \text{3rd approx. } \frac{39}{16}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{(\tfrac{8}{3})}}}} \qquad = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + (\tfrac{2}{3})}}}} \qquad \text{4th approx. } \frac{95}{39}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{(\tfrac{3}{2})}}}}} \qquad = 2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}}}} \qquad \text{5th approx. } \frac{134}{55}.$$

EXERCISE LXVII.

☞ Convert each of the following fractions into a continuous form, and find the series of convergents by the method described in (235).

1. $\frac{829}{347}$; $\frac{159}{493}$.
2. $\frac{1104}{887}$; $\frac{173}{200}$.
3. $\frac{425}{1717}$; $\frac{603}{138}$.
4. $\frac{113}{764}$; $\frac{456}{121}$.
5. $\frac{2602}{58363}$; $\frac{503}{100103}$.
6. $\frac{211}{417}$; $\frac{718}{125}$.
7. $\frac{1845}{1717}$; $\frac{427}{365}$.
8. $\frac{68}{99}$; $\frac{65}{149}$.
9. $\frac{351}{965}$; $\frac{251}{764}$.
10. $\frac{1769}{5537}$; $\frac{907}{18564}$.
11. $\frac{1947}{3359}$; $\frac{587}{1943}$.
12. $\frac{2456}{66922}$; $\frac{5743}{80937}$.

SECTION VII.—MISCELLANEOUS APPLICATIONS OF VULGAR FRACTIONS.

237. In many parts of arithmetic, fractional expressions occur, which require the foregoing rules to solve them. Such cases assume very varied forms, and are generally comprehended under one or other of the following cases.

238. CASE I.—WHEN THE VALUE OF THE WHOLE QUANTITY IS GIVEN, TO FIND THE NEAREST INTEGRAL VALUE FOR A FRACTION.

RULE.

Multiply the quantity by the numerator, and divide by the denominator.

Example.—What is the value of $\frac{5}{8}$ of £1? Here $\frac{1}{8}$ of £1 = 2s. 6d. Therefore $\frac{5}{8}$ of £1 must equal 5 times 2s. 6d., or 12s. 6d. Or, because $\frac{5}{8}$ of £1 = $\frac{1}{8}$ of £5, therefore £5 ÷ 8, or 12s. 6d., is the answer.

In either case it will be seen that the £1 is multiplied by 5 and divided by 8, Whether we divide first and then multiply, or multiply first and divide afterwards, the answer is the same.

239. *Observation.*—Suppose it is required to find $\frac{7}{11}$ of 200. If we try to take $\frac{1}{11}$ of 200, and then multiply by 7, we first obtain $200 \div 11 = 18\frac{2}{11}$, and this multiplied by 7 gives $126\frac{14}{11}$, or $127\frac{3}{11}$. As we have had to deal with fractions in both processes, it would have been simpler to multiply the 200 by 7 in the first instance, and then divide this product by 11; for $\frac{7}{11}$ of 200 is the same as $\frac{1}{11}$ of 7×200, or $\frac{1400}{11} = 127\frac{3}{11}$.

So if it be required to find $\frac{8}{9}$ of 2s. 6d., this may be found in two ways—I., by taking $\frac{1}{9}$ of 2s. 6d. and multiplying it by 8; and II., by taking 8 times 2s. 6d. and dividing it by 9.

	s.	d.			s.	d.
I. 9)	2	6		II.	2	6
		$3\frac{1}{4} + \frac{1}{3}$ of a farthing				8
		8		9)	20	0
	2	$2\frac{1}{2} + \frac{2}{3}$ of a farthing			2	$2\frac{1}{2} + \frac{2}{3}$ of a farthing.

The second of these two methods is preferable, because by it the fractional part of the answer is obtained with less trouble. Hence it should be remembered—

In all cases in which it is required to multiply by one number and divide by another, it is more convenient to perform the multiplication first and the division afterwards.

EXERCISE LXVIII.

1. Find $\frac{7}{8}$ of 2s. 7½d.; $\frac{14}{15}$ of $\frac{3}{4}$ of £5.
2. $\frac{6}{7}$ of 1 cwt.; $\frac{3}{5}$ of $\frac{4}{9}$ of a ton.
3. $\frac{2}{3}$ of $\frac{7}{9}$ of the product of 12 and 9; $\frac{14}{25}$ of $5\frac{1}{2}$ miles.
4. $\frac{7}{15}$ of $\frac{12}{19}$ of £50; $\frac{18}{23}$ of £7 3s. 4d.
5. $\frac{7}{8}$ of £4 3s. 9d.; $\frac{7}{12}$ of £18 6s. 8d.
6. $\frac{4}{5}$ of $\frac{20}{19}$ of £81 5s. 4d.; $\frac{2}{7}$ of $\frac{3}{5}$ of £75 10s.
7. $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{8}$ of 20 guineas; $\frac{5}{7}$ of $\frac{3}{14}$ of $\frac{8}{9}$ of £205.
8. $5\frac{4}{9}$ of a hogshead of ale; $17\frac{2}{3}$ of a hogshead of wine.
9. $\frac{4}{7}$ of 2 miles; $\frac{13}{15}$ of an acre; $\frac{7}{12}$ of a square mile.
10. $5\frac{7}{8}$ of a bushel; $\frac{4}{17}$ of a quarter; $12\frac{5}{7}$ of a gallon.
11. $\frac{4}{7}$ of $\frac{2}{19}$ of £20; $\frac{8}{53}$ of £11; $\frac{2}{7}$ of $\frac{5}{9}$ of a florin.
12. $\frac{5}{7}$ of a lunar month; $\frac{2}{13}$ of a year.
13. $\frac{6}{7}$ of $\frac{3}{5}$ of 4 miles; $\frac{4}{11}$ of $\frac{11}{17}$ of a league.
14. $\frac{3}{5}$ of $\frac{5}{17}$ of 2 tons; $\frac{7}{10}$ of $\frac{2}{11}$ of 4 cwt.
15. Add together $\frac{7}{11}$ of £1, $\frac{2}{13}$ of a crown, and $\frac{4}{17}$ of a guinea.
16. From $\frac{2}{3}$ of a square mile take $17\frac{2}{3}$ acres.
17. By how much does $\frac{3}{8}$ of a sovereign exceed $3\frac{1}{4}$ Prussian thalers, the value of a thaler being three shillings.
18. What proportion is $\frac{3}{4}$ of a yard of half a mile?

240. CASE II.—WHEN THE VALUE OF A FRACTION IS GIVEN, TO FIND THE WHOLE QUANTITY OF WHICH IT IS THE FRACTION.

RULE.

Multiply the given value by the denominator and divide by the numerator.

Example I.—What is the number of which 30 is $\frac{5}{7}$? Here if 30 is $\frac{5}{7}$ of the required number, the fifth part of 30 must be $\frac{1}{7}$ of that number. But the fifth of 30 = 6; wherefore 6 is $\frac{1}{7}$ of the answer. But 6 is $\frac{1}{7}$ of 6 × 7 or of 42; therefore 42 is the answer, for 30 is $\frac{5}{7}$ of 42.

Example II.—What is the number of which x is $\frac{a}{b}$? Here it is obvious that the whole number required must be as much greater or less than x as b is greater or less than a. Wherefore $\frac{x\,b}{a}$ = the answer required.

Example III.—What is the sum of money of which 5s. is $\frac{10}{17}$? Here if 5s. is ten seventeenths of the required sum, a tenth of 5s. must be one seventeenth. But 5s. ÷ 10 = 6d.; therefore 6d. = $\frac{1}{17}$ of the required sum. But 6d. × 17 = 102d. = 8s. 6d.; wherefore 8s. 6d. is the sum of which 5s. is $\frac{10}{17}$.

EXERCISE LXIX.

1. What is the period of which 3 hours 20 mins. is $\frac{5}{8}$?: Of which 3 days is $\frac{6}{7}$?
2. What sum is that of which 3s. 6d. is $\frac{14}{17}$: Of which £7 12s. 6d. is $\frac{5}{9}$?
3. What length is that of which 25 yards is $\frac{15}{17}$?: Of which 7 feet is $\frac{21}{100}$?
4. Of what weight is 2 oz. 3 dwt. three seventeenths?
5. What weight is that of which 17 lbs. 2 oz. is $\frac{12}{13}$?: Of which 5 cwt. 2 qrs. is $\frac{9}{11}$?
6. Of what number is 17 equal to $\frac{15}{23}$?: Of what is $5\frac{1}{3}$ equal to $\frac{25}{34}$?
7. What is the whole of which £5 12s. 6d. is $\frac{3}{8}$?
8. 17 men have equal allotments of land in a field, the space of five of them amounts to 7 roods 20 perches; what is the area of the field?
9. What amount is that of which the sum of our English silver coins is $\frac{3}{7}$?
10. What period is that of which $\frac{2}{15}$ is equal to $2\frac{1}{3}$ lunar months?

241. CASE III.—TO EXPRESS A GIVEN QUANTITY AS A FRACTION OF ANOTHER OF THE SAME KIND.

RULE.

Reduce both to the same name; the number contained in the given quantity, and that in the other of the same kind, will be the numerator and denominator of the fraction required.

Example I.—What fraction is 20 of 29? According to the Definition of Fractions (182) 20 is $\frac{20}{29}$ of 29. Questions of this kind, in which abstract numbers only are concerned, can readily be solved by making one of the numbers a numerator and the other a denominator; for, whether a be greater or less than b, it is always true that $\frac{a}{b}$ represents the fraction a is of b.

But when the numbers concerned are concrete this rule cannot be adopted unless they refer to the same name. Thus the fraction 13 feet is of 20 yards is not $\frac{13}{20}$. On reducing the yards to 60 feet, the question becomes "What fraction is 13 feet of 60 feet?" and the answer is clearly $\frac{13}{60}$.

Example II.—What fraction of 15s. $9\frac{1}{2}$d. is $2\frac{3}{4}$d.?

By rule of Descending reduction (87) 15s. $9\frac{1}{2}$d. = 758 farthings, and $2\frac{3}{4}$d. = 11 farthings. Hence $2\frac{3}{4}$d. = $\frac{11}{758}$s of 15s. $9\frac{1}{2}$d.

EXERCISE LXX.

1. What fraction of $4\frac{1}{2}$ yards is $3\frac{1}{4}$ inches?
2. What fraction of a guinea is 3s. $4\frac{1}{2}$d.?
3. What fraction of a cwt. is 2 lbs. $6\frac{1}{2}$ oz.?
4. What fraction of 5 tons is 3 cwt. 2 qrs. 17 lbs.?
5. What fraction of 5 weeks is 2 hours 20 mins.?
6. Express £17 3s. 10d. and £2 3s. 8d. as fractions of £100.
7. Express 2 gallons 1 pint as a fraction of a barrel of beer.
8. Express 3s. $4\frac{1}{4}$d. as fractions of a crown and a guinea.
9. If a mountain be $4\frac{7}{8}$ miles high, express its altitude as a fraction of the earth's diameter, which is 7926 miles.
10. A parish contains 7,233 acres 29 poles; express its area as a fraction of the whole of England, which contains 58,000 sq. miles.
11. Express fractionally the proportion that the United Kingdom, which consists of 119,000 square miles, bears to Europe, which contains 3,900,000.

PRACTICE.

242. In this Rule it is required to find by the help of fractions the value of any number of articles when the price of one is known.

Observation.—In (138) it was shown that if a number increased any number of times made up another, the first was called a *measure* of the second, and the second a *multiple* of the first. But these terms are seldom used except in the case of abstract numbers. When any *concrete* number taken a certain number of times makes up another, the first is commonly called an *Aliquot part* of the second. Thus 5s. is an aliquot part of £1; 14 lbs. is an aliquot part of a hundredweight; 2 ounces of a pound. But 3s. is not an aliquot part of sovereign; nor is 7 ounces an aliquot part of a pound.

Example I.—Suppose it is required to find the price of 2834 articles at 17s. 10½d. each.

Now 2834 articles at £1 each would be worth £2834.

	s.	d.		£	s.	d.		£	s.	d
Cost of 2834 articles at	10	0	each, or ½ of	2834	0	0	=	1417	0	0
Cost of 2834 articles at	5	0	each, or ½ of	1417	0	0	=	708	10	0
Cost of 2834 articles at	2	6	each, or ½ of	708	10	0	=	354	5	0
Cost of 2834 articles at		3	each, or $\frac{1}{10}$ of	354	5	0	=	35	8	6
Cost of 2834 articles at		1½	each, or ½ of	35	8	6	=	17	14	3
Cost of 2834 articles at	17	10½	each				=	2532	17	9

or the sum of the several answers.

The same sum might have been worked thus:—

d.		s.	d.	
6	2	2834		= cost of 2834 articles at 1s. each.
		17		
		48178		= cost at 17s.
3	2	1417		= cost at 6d., or half the cost at 1s.
1½	2	708	6	= cost at 3d., or half the cost at 6d.
		354	3	= cost at 1½d., or half the cost at 3d.
		20) 50657	9	= cost at 17s. 10½d.
		£2532 17	9	Answer reduced to pounds.

Or, if 2834 pence were taken as the standard, the number might have been multiplied by the number of pence in 17s. 10d. Then half the upper line might have been taken to give the value of the articles at one halfpenny, and with this added, the answer would appear in pence.

243. No rule can be given for determining what "aliquot" part should be chosen. A little practice will soon enable a learner to select the most convenient. Three things only require to be remembered :

I. The number of articles given may be taken to represent the number of pounds, shillings, pence, or farthings, which so many articles would cost at a pound or shilling, a penny or a farthing each.

II. This number must be multiplied or divided according as the answer required is to be greater or less.

III. Each line will represent the cost of the given number of articles at a certain part of the price each, and the sum of all the lines will represent the cost of the given number of articles at the whole of the price each, and will give the answer.

RULE FOR PRACTICE.

244. Multiply the number of articles by the number of pounds or shillings in the price, and take aliquot parts for the rest.

Example.—Find the value of 8632 articles at £1 14s. 3¾d. each.

	£	*s.*	*d.*	
10s. = £1 ÷ 2	8632	0	0	= cost at £1 each
2s. 6d. = 10s. ÷ 4	4316	0	0	= cost at 10s. each
1s. 3d. = 2s. 6d. ÷ 2	1079	0	0	= cost at 2s. 6d. each
	539	10	0	= cost at 1s. 3d. each
6d. = 2s. 6d. ÷ 5	215	16	0	= cost at 6d. each*
¾d. = 6d. ÷ 8	26	19	6	= cost at ¾d. each
	14809	5	6	= cost at £1 14s. 3¾d.

EXERCISE LXXI.

1. 7246 articles at £7 8s. 9d.; 2397 at £2 6s. 8¾d.
2. 4096 at £17 3s. 5d.; 50832 at 4s. 3¼d.
3. 20738 at £6 6s. 0¾d.; 10096 at 17s. 2¼d.
4. 1309 at £25 4s. 7d.; 70862 at £1 12s. 6¾d.
5. 20891 at £14 2s. 11¼d.; 35619 at £7 13s. 3½d.,
6. 1057 at 10s. 6d.; 20731 at 15s. 6d.
7. 16379 at 5s. 4¼d.; 41986 at £1 10s. 4¼d.

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* Observe that this line has not been obtained from that immediately above it, but from the preceding, because 6d. is an aliquot part of 2s. 6d., but not of 1s. 3d.

8. 12357 at 11¾d.; 32705 at 6s. 7½d.
9. 1178 at £5 6s. 3¼d.; 20196 at £15 3s. 5d.
10. 36917 at 7s. 2¼d.; 51398 at 15s. 10d.
11. 37128 at 14s. 6½d.; 18603 at 17s. 11½d.
12. 41357 at £53 7s. 3¼d.; 2198 at £16 2s. 7d.
13. 218763 at 10¼d.; 41378 at 7¾d.
14. 71639 at £4 17s. 2d.; 21086 at £3 12s. 3d.
15. 10397 at £5 6s. 3d.; 2178 at 17s. 10½d.
16. 3147 at 1s. 9½d.; 41263 at 14s. 6¼d.
17. 21076 at £15 11s. 2½d.; 1413 at £12 10s. 7¼d.
18. 17621 at £1 16s. 3¾d.; 2108 at £1 6s. 2¼d.
19. 30128 at £19 4s. 2d.; 17186 at £12 13s. 8d.
20. 10719 at £43 2s. 5d.; 20165 at £57 19s. 4¼d.
21. What is the value of 23,654 articles at £17 15s. 6½d. each?
22. In an army the number of horses is 23,456, which cost on an average £44 16s. 10½d. each; what is the value of the whole?

245. When the number of times the price has to be taken is not simply expressed, but is shown by numbers of different denominations, the question is said to be in COMPOUND PRACTICE.

Example I.—What is the value of 2 cwt. 3 qrs. 16 lbs. at £14 7s. 6d. per cwt.?

	£.	s.	d.	
2 qrs. = 1 cwt. ÷ 2	14	7	6	= value of 1 cwt.
			2	
	28	15	0	= value of 2 cwt.
1 qr. = 2 qrs. ÷ 2	7	3	9	= value of 2 qrs.
14 lbs. = 1 qr. ÷ 2	3	11	10½	= value of 1 qr.
2 lbs. = 14 lbs. ÷ 7	1	15	11¾	= value of 14 lbs.
		5	$1\frac{19}{28}$	= value of 2 lbs.
	41	11	$8\frac{26}{28}$	= value of 2 cwt. 3 qrs. 16 lbs.

Example II.—How much must be paid for 27 acres 3 roods 27 poles at £7 10s. 8d. per acre?

	£	s.	d.	
2 roods = ½	7	10	8	= price of 1 acre
			9	
	67	16	0	= price of 9 acres
			3	
	203	8	0	= price of 27 acres
1 rood = ½	3	15	4	= price of 2 roods
20 poles = ½	1	17	8	= price of 1 rood
5 poles = ¼		18	10	= price of 20 poles
2 poles = $\frac{1}{10}$ of 20 poles		4	8½	= price of 5 poles
		1	$10\frac{6}{10}$	= price of 2 poles
	210	6	$5\frac{1}{10}$	= price of 27 a. 3 r. 27 p.

EXERCISE LXXII.

1. 2 cwt. 3 qrs. 10 lbs. at £1 16s. 6d. per cwt.

2. 17 cwt. 1 qr. 15 lbs. 7 oz. at £2 15s. 3d. per cwt.

3. 3 tons 14 cwt. 11 lbs. at £7 3s. 8d. per cwt.

4. 3 qrs. 17 lbs. 9 oz. at £10 per cwt.

5. 5 acres 2 roods 27 poles at £5 10s. per acre.

6. 27 acres 1 rood 9 poles at £7 12s. per acre

7. 13 acres 3 roods 33 poles at £11 per acre.

8. 200 acres 2 roods 17 poles at 11s. 6d. per rood.

9. 18 lbs. 13 oz. at £100 per ton.

10. 7 gallons 3 pints at £1 7s. per barrel.

11. 3 quarts 1½ pint at £2 10s. per butt.

12. 1 bush. 3 pecks at £2 18s. per quarter.

13. Find the wages for 17 months 3 weeks 6 days at £2 10s. 6d. per month.

14. Rent of 246 acres 2 roods 15 poles at £1 7s. 6d. per acre.

15. 15 cwt. 2 qrs. 17 lbs. at £2 6s. 7d. per quarter.

16. 17 cwt. 1 qr. 9 lbs. 6 oz. at 1s. 8½d. per lb.

17. What is the rent of an estate of 656 acres 3 roods 14 poles, which is let at £1 13s. 8d. per acre?

18. If sugar costs £2 17s. 6d. per cwt., find the value of 19 tons 17 cwt. 21 lbs.

19. The planting of a rood of ground cost £27 18s. 6d.; what was paid for planting 23 acres 3 roods 25 poles at the same rate?

20. If the erection of a fence costs £215 8s. 4d. per mile, what will it cost to enclose a park whose boundary measures 17 miles 3 furlongs 118 yards?

21. In a French theatre, there were 120 persons in the boxes, 275 in the pit, and 348 in the gallery. The prices of admission were 2½ francs for the boxes, 1½ franc for the pit, and 70 centimes for the gallery. The lighting cost 45 francs 80 centimes, the music 50 francs, and incidental expenses 65 francs. What were the net receipts? (100 centimes = 1 franc).

22. Compare the values of 777 articles at £3 7s 10¾d. each, and 923 articles at £2 15s. 10¼d. each.

Questions on Vulgar Fractions.

Distinguish between Integral and Fractional Arithmetic. To which of the former rules may Fractional Arithmetic be considered the sequel? Why? Give the meaning of the words, fraction, integer, numerator, denominator, proper, and improper.

What is a mixed number, and what change may it always undergo? Give an example, and state the reason. How may a fraction be multiplied or divided by an integer? Which is the preferable method of dividing $\frac{2}{3}$ by 2, and why?

What changes may every fractional expression undergo State the principle, and give a demonstrative example. What is the effect on a fraction of equal increase or diminution to its terms, and why? Suppose $\frac{10}{30}$ be converted into $\frac{1}{3}$, what advantage is gained, and what principle is illustrated? Suppose $\frac{5}{7}$ is converted into $\frac{15}{21}$, what advantage is gained, and what principle illustrated?

In what rule of Fractions are the rules for finding the greatest common measure and the least common multiple useful, and why? When does the addition of the numerators effect the addition of the fractions? Give the rule for finding a common denominator, and state when and why it is to be used. What truths are assumed in the process of adding $\frac{5}{7}$ to $\frac{3}{8}$? Demonstrate each step of the process.

Define multiplication in the widest sense of the term. How does Fractional Multiplication differ from Integral? In how many ways can you prove that $\frac{4}{9} \times \frac{3}{5} = \frac{4 \times 3}{9 \times 5}$? Demonstrate the rule by each method. Define Division, and show how Fractional differs from Integral Division. What is a reciprocal? In how many ways can you prove that $\frac{4}{9} \div \frac{3}{5} = \frac{4 \times 5}{9 \times 3}$? Demonstrate the rule by each method.

What is the general rule for reducing a fraction to another of a different denominator? Suppose $a = nb$, what fraction of a is $\frac{1}{3}$ of b, and what fraction of b is $\frac{2}{3}$ of a? Give the reason in each case.

Explain why, in a series of convergent fractions, the several approximations are alternately greater and less than the fraction itself.

Why is it better to multiply first and divide afterwards, when both have to be done in one sum? In what operation should we divide a quantity by the numerator, and multiply by the denominator, and when should the inverse process be used? Give the reason in both cases. What is the general rule to be observed in Practice, and how is it connected with Fractions?

State in words the truth concerning fractions illustrated by each of the following formulæ:—

1. $\frac{a}{b} \times c = \frac{ac}{b} = \frac{a}{b + c}$ 2. $\frac{a}{b} \div c = \frac{a \div c}{b} = \frac{a}{bc}$

3. $\frac{a}{b} = \frac{ma}{mb} = \frac{a \div m}{b \div m}$ 4. $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$

5. $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd}$ 6. $a + \frac{b}{c} = \frac{ac + b}{c}$

7. If x be less than y, $\frac{x}{y}$ is less than $\frac{x + a}{y + a}$, but greater than $\frac{x - a}{y - a}$

If x be greater than y, $\frac{x}{y}$ is greater than $\frac{x + a}{y + a}$, but less than $\frac{x - a}{y - a}$

8. $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ 9. $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{cb}$

10. $\frac{a}{b} = \frac{1}{b \div a}$ 11. $\frac{a}{b} \div \frac{c}{b} = a \div c.$

General Exercises on Vulgar Fractions.

1. Add together $\frac{5}{9}$ of £1, $\frac{3}{14}$ of a guinea, $\frac{2}{11}$ of a crown, and $\frac{7}{8}$ of a shilling.

2. If 2 men can do a certain work together in 15 days, and one of them could do it by himself in 25 days, how long would the other be in doing it alone?

3. Find the difference between the square of $4\frac{7}{8}$ and the cube of $2\frac{1}{3}$.

4. What fraction of £7 is equivalent to $\frac{7}{8}$ of a guinea?

5. Express the area of a plot of 3 roods 7 poles, as a fraction of a field of $9\frac{1}{2}$ acres.

6. Of how many pounds of sugar does a tradesman defraud his customers in retailing $4\frac{3}{7}$ cwt. of sugar, if he uses a false weight of $15\frac{5}{8}$ oz. for a pound?

7. What number divided by $\frac{3}{5}$ will give $3\frac{7}{8}$ as quotient? and what number multiplied by $13\frac{4}{7}$ will give $15\frac{9}{13}$ as product?

8. Find the sum of $\frac{2}{3}$, $\frac{1}{7}$, and $\frac{2}{5}$, and the product of $2\frac{1}{3}$ and $3\frac{2}{5}$.

9. Find the difference between the sum of $\frac{2}{3}$ and $\frac{4}{9}$, and the product of $\frac{7}{12}$ and $\frac{8}{11}$.

10. Express the difference between 12 and $11\frac{4}{73}$ as a converging fraction.

11. What is the value of 19 yards 2 feet 5 inches at 5s. 7d. per foot?

12. Find the land tax on 43 acres 2 roods 26 poles at 3s. 7d. per acre.

13. How much of £5 8s. 4d. is $\frac{2}{3}$ of $\frac{1}{8}$ of £20?

14. Divide the product of $4\frac{7}{8}$ and $5\frac{4}{9}$ successively by their sum and their difference.

15. From the sum of $\frac{2}{7}$ and $\frac{4}{17}$ take the difference between $\frac{3}{8}$ and $\frac{5}{11}$.

16. Which is greater, the product of $17\frac{2}{7}$ and $5\frac{3}{8}$, or the product of $\frac{72}{5}$ and $12\frac{2}{3}$, and what is the difference?

17. Find the difference between the continued product of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, and that of $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, and $\frac{7}{8}$.

18. What is the product of $\frac{2}{3}$ of $\frac{7}{8}$ and $\frac{5}{9}$ of $8\frac{1}{4}$?

19. A boy has a number of marbles, of which he loses $\frac{3}{8}$ at play, gives away $\frac{5}{16}$ to one schoolfellow, and $\frac{1}{4}$ to another; he then has seven left: how many had he at first?

20. A father leaves to one of his children $\frac{1}{5}$ of his property, to another £200, and the remaining $\frac{3}{5}$ among the rest; how much does he leave altogether?

21. Add together $\frac{17}{53}$ of £176 18s., and $\frac{11}{18}$ of £150.

22. If a post be $\frac{1}{3}$ in the water, $\frac{1}{7}$ out of the water, and 22 feet in the mud, what is its length?

23. How much must be added to $3\frac{4}{11}$ to make $7\frac{2}{15}$? and what number taken from $23\frac{5}{18}$ will leave $\frac{5}{7}$ of $9\frac{1}{4}$?

24. What is the value of $\frac{28}{67}$ of a ship, $\frac{5}{6}$ of which is valued at £755 15s.?

25. Divide the difference between $\frac{7}{8}$ and $\frac{5}{14}$ by the sum of $\frac{1}{7}$ and $\frac{5}{6}$.

26. If of a tavern bill of £3 0s. $11\frac{1}{2}$d. my share and that of two others amounts to 16s. $7\frac{1}{2}$d., how many are there in the company.

27. If the area of a table be 76 square inches, and its length $10\frac{5}{8}$ inches, what is its breadth?

28. After spending $\frac{2}{7}$ of the money in my purse I find that $\frac{3}{4}$ of the remainder amounts to 16s. 3d., how much had I at first?

29. Subtract from 5 its third, its fourth, and its fifth; what fraction of $17\frac{1}{4}$ is the remainder?

30. If the product of $\frac{7}{8}$ and $2\frac{1}{4}$ be added to the sum of $21\frac{5}{7}$ and $3\frac{6}{11}$, by how much will the result differ from 100?

31. Compare the magnitudes of $\frac{11}{19}$, $\frac{18}{29}$, and $\frac{28}{38}$.

32. Compare the values of $\frac{1}{19}$ of £1, $\frac{1}{20}$ of a guinea, and $\frac{8}{35}$ of a crown.

33. What is the cube of that number which, when multiplied by $\frac{3}{8}$ of $\frac{4}{5}$ of $1\frac{1}{2}$, will produce 1?

34. What is the net annual income of a man whose estate is worth £1023 10s. per annum, but who pays as land tax 2s. $8\frac{1}{2}$d. in the pound?

35. If the owner of $\frac{4}{17}$ of a ship sold $\frac{8}{11}$ of $\frac{2}{9}$ of his share for £$\frac{400}{35}$, what was the value of $\frac{1\frac{1}{4}}{4\frac{3}{8}}$ of $\frac{2}{5}$ of the whole ship at the same rate?

36. If two-thirds of a business be worth £440, what is the value of $\frac{3}{11}$ of it?

37. If $\frac{7}{15}$ of a guinea be taken from $\frac{2}{13}$ of $\frac{5}{9}$ of a five pound note, what fraction of £100 will remain?

38. A owns $\frac{9}{16}$ of a field, and B the rest; the difference between their allotments is 3 roods 17 poles; what is the area of the field?

39. In Fahrenheit's thermometer the range between freezing water and boiling water is divided into 180°, while in Réaumur's it is only 80°, and in the Centigrade 100°; how would a change of temperature of 18° Fahrenheit be marked on the Réaumur and Centigrade instruments, and what would $6\frac{5}{12}$° of Réaumur correspond to in the Centigrade and Fahrenheit thermometers?

SECTION VIII.—DECIMAL FRACTIONS.

246. **A Fraction is called decimal when its denominator is either ten, one hundred, one thousand, or some power* of ten.**

Thus $\frac{3}{10}$, $\frac{5}{100}$, $\frac{7}{1000}$, $\frac{183}{100000}$ are Decimal Fractions.

Two or three facts concerning the principle of our Notation require to be distinctly recalled here.

I. In Integral Arithmetic the number 10 is the uniform instrument of multiplication, and we deal with all the collections of numbers which are brought before us as composed of tens and of powers of ten. Thus, if we have to think of $6 + 9 + 8$ we instantly resolve it into 23, or *two* TENS and 3; or if the number 5 times 7 is spoken of, we do not deal with it until it is transformed into 35, *i. e., three tens* and 5.

II. Every figure in the following line means 10 times more than that on its right; the last figure alone meaning 7, and all the rest having a higher value.

7 7 7 7 7

Here the first 7 signifies 7 times the fourth power of 10, the second 7×10^3, the third $= 7 \times 10^2$, and the fourth 7×10.

247. The object of all Integral Arithmetic is to express every collection of numbers whatever,—their sums, their differences, their products or their quotients, *in the form of multiples, either of ten or of some power of ten.*

248. **Decimal Fractions are designed to express all *fractions* whatever, *i. e.*, all divisions of units into parts, as divisions by ten or by some power of ten.**

249. *Observation.*—If we have to express 9 sevens we write down the multiplier thus, 9×7. But to express 9 tens it would not be necessary to write the 10, for by placing the 9 with one figure to its right, as 90 or 96, the multiplication is at once effected. So also if we mean $\frac{7}{9}$, or 7 divided by 9, we have to write down the divisor 9; but if we wish to deal with $\frac{7}{10}$ or 7 divided by 10, the decimal system enables us to express this without writing the 10.

---

* By *power* of a number we mean the product of equal factors:—

e.g., $5 \times 5 = 5^2$ = the second power of five.

$5 \times 5 \times 5 = 5^3$ = the third power of five.

$5 \times 5 \times 5 \times 5 = 5^4$ = the fourth power of five.

250. In (246) each of the figures 77777 represented a tenth part of that on its left. In Fractions we can extend this arrangement below unity, still letting each figure mean a tenth of that on its left. In the following line let the figure with the line above it mean units.

$$\overset{5}{7}\ \overset{4}{7}\ \overset{3}{7}\ \overset{2}{7}\ \overset{1}{7}\ \overline{7}\ \overset{1}{7}\ \overset{2}{7}\ \overset{3}{7}\ \overset{4}{7}\ \overset{5}{7}$$

Then the figure to its left is 7 × 10 or 70, but that on its right is $\frac{7}{10}$ or $7 \div 10$. So also the figure in the second place to the left of the unit = 700 or 7×10^2, but that two places to the right = $\frac{7}{100}$ or $7 \div 10^2$. The next pair of corresponding figures to the left and right mean respectively 7000 or 7×10^3, and $\frac{7}{1000}$ or $7 \div 10^3$. The next two are 70000 or 7×10^4, and $\frac{7}{10000}$ or $7 \div 10^4$.

Here the middle figure alone refers to unity, and not the last, as is the case in integer numbers, yet the same principle of notation still applies, viz., *that the value of the numbers increases ten times at each place to the left,* and that each figure represents $\frac{1}{10}$ of the value it would have if it were one step further to the left.

251. Instead of placing a mark over the unit as we have done, it is usual to place a point (.) between the unit and the first fractional number, thus:—

568340·24739 means

hundred thousands.	ten thousands.	thousands.	hundreds.	tens.	units.		tenths.	hundredths.	thousandths.	ten thousandths.	hundred thousandths.
5	6	8	3	4	0	.	2	4	7	3	9

$$578{\cdot}324 = (5 \times 100) + (7 \times 10) + 8 + \tfrac{3}{10} + \tfrac{2}{100} + \tfrac{4}{1000}.$$

$$70{\cdot}069372 = (7 \times 10) + \tfrac{6}{100} + \tfrac{9}{1000} + \tfrac{3}{10000} + \tfrac{7}{100000} + \tfrac{2}{1000000}$$

Exercise LXXIII.

☞ Write out the separate value of each figure in the following lines:—

1. 2970·32604; 3·087.
2. 14·109; ·604.
3. 1·684; ·052.
4. 103·561; 75·1086
5. 3271·0986; 176·987.
6. 12·3456; 1·23456.
7. 507·1862; 90·006.
8. 12·037; 713·586.

252. In Decimal Fractions the denominator is not usually expressed.

The purpose of the denominator is to show the value of *each part*, of which the numerator shows the *number*. As in a decimal fraction this is always 1 with as many ciphers to its right as the numerator has figures, it is useless to write it down.

Observation.—In writing integer numbers (11) there is always one place (that on the right) reserved for the unit, and if there is no unit the place is filled by a cipher. But in writing decimal fractions it is not necessary to reserve a place for the unit, as the first figure to the right of the decimal point always means tenths. Thus the two arrangements do not at first sight appear to correspond exactly, for in integers the *second* figure in the series means tens, while in fractions the *first* figure means tenths; so also the third figure to the left of the point means hundreds, while the *second* figure to its right means hundredths. This will be seen in the following cases—

$$3640 = (4 \times 10), + (6 \times 10^2), (3 \times 10^3) \text{ and } \cdot 463 = \frac{4}{10}, + \frac{6}{10^2}, + \frac{3}{10^3},$$

Although these two expressions correspond exactly in meaning, the one being above unity and formed by decimal multiplication, and the other below unity and formed by decimal division, yet four figures are needed to express the first, and only three to express the second. Nevertheless, it must be remembered, that in all expressions which are partly integral and partly fractional, the figures at equal distances from the unit correspond in value:—

$$7\quad 3\quad 2\quad 5\ \cdot\ 6\quad 4\quad 9$$

Here the 2 and the 6, which are equally distant from 5, correspond in value, the first meaning tens and the second tenths. So the 3 means three hundreds and the 4 four hundredths, each being in the third place from the unit. The language employed in (15) will therefore still be available, tens and tenths being both units of the second place, thousands and thousandths being both units of the fourth place, millions and millionths being both units of the seventh place. When the fraction stands alone without an integer the point itself must be counted as one, and the first figure (tenths) as being a unit of the second place, and then the language of Integral Notation will still apply.

253. *In Integer numbers a cipher to the right of a figure increases its value ten times, but a cipher to its left leaves its unaltered. In Fractional numbers a cipher to the left of a number diminishes its value ten times, but a cipher to the right leaves it unaltered.*

254. The cipher is simply intended (16) to put any given figure in its right place with regard to the unit. In integer numbers, if a cipher

does not stand to the right of a figure it does not affect its value at all. Thus, 057 means the same as 57, the 0 has no meaning, for it does not stand between any figure and the unit; but if the number were 2057 the 0 would affect the value of the 2 by placing it a step further from the unit, although it would not affect the value of the 5 or the 7. Similarly, ciphers have a meaning in Decimal Fractions when they place a number further from the unit, but not otherwise. In the expression ·57, the 5 means $\frac{5}{10}$ and the 7 $\frac{7}{100}$; but in the expression ·5700 they still retain the same value, and the ciphers, because they do not affect the position of either the 5 or the 7 with regard to the unit, are without meaning. But in ·057 the cipher really affects the value of both, for it makes the 5 mean $\frac{5}{100}$ instead of $\frac{5}{10}$, and the 7 $\frac{7}{1000}$ instead of $\frac{7}{100}$.

255. *Corollary.—In any line of figures, to move the decimal point one place to the right is to multiply the whole line by* 10, *but to remove the point one place to the left is to divide the whole line by* 10. *To remove the point two places is to multiply or divide by a hundred; three places by a thousand.*

Example.—(a). 1783·964. (b). 178·3964. (c). 17839·64.

In (a) the figure 3 is the unit, and every figure is greater or less according to its distance from that unit; but in (b) the decimal point having been removed one place to the left, the 8 which meant 8 tens becomes 8 only; the 3 which meant 3 units becomes the 10th part of 3, and every figure in (b) represents the tenth part of the value it had in (a). Now, on comparing (a) and (c), we find that the opposite effect has been produced by putting the point one place to the right. For in (c) the 3 means 3 tens instead of 3 units; the 8 means 8 hundreds instead of 8 tens, and every figure means ten times what it means in (a). By comparing (b) and (c) we find that every figure in the latter means a hundred times its value in the former, because the decimal point is two places further to the right.

Exercise LXXIV.

☞ Multiply the following expressions:—

1. By ten. 1·078; 56·320; ·079; 513·087; 16·54.
2. By a hundred. 235·96; 50·985; ·10372; ·001; 1·8739.
3. By ten thousand. 41·0968; 1·50796; ·00183; 723·568701.

Divide the following numbers:—

1. By ten. 35·607; ·178; 4·762; ·0018; 3·729.
2. By a thousand. 13·569; 27·48; 47·2; 3·069; 18372·1.
3. By a million. 27486·32; 5098·527; 18·07629; 453786·1.

256. Decimals have this advantage over vulgar fractions, that they can be instantly reduced to a common denominator, and can therefore be added, subtracted, or compared without trouble.

Example.—The figures 792·8345 taken separately represent—

$(7 \times 100) + (9 \times 10) + 2 + \frac{8}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}$.

10000 is here the greatest denominator in the series, and is also a common multiple of all the other denominators. Take the fraction $\frac{4}{1000}$, multiply its terms by 10 and it becomes $\frac{40}{10000}$; the former fraction, in like manner, becomes $\frac{300}{10000}$; $\frac{8}{10}$ becomes $\frac{8000}{10000}$, and so on.

Thus $700 = \frac{7000000}{10000}$, $90 = \frac{900000}{10000}$, $2 = \frac{20000}{10000}$, $\frac{8}{10} = \frac{8000}{10000}$, $\frac{3}{100} = \frac{300}{10000}$, and $\frac{4}{1000} = \frac{40}{10000}$.

$$792{\cdot}8345 = \frac{7000000 + 900000 + 20000 + 8000 + 300 + 40 + 5}{10000} = \frac{7928345}{10000}.$$

A separate analysis of the parts of this expression gives the following results :—

$792{\cdot}8345 = 700 + 90 + 2 + \frac{8}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}$.
$792{\cdot}8345 = 792 + \frac{8}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}$.
$792{\cdot}8345 = \frac{7928}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000}$.
$792{\cdot}8345 = \frac{79283}{100} + \frac{4}{1000} + \frac{5}{10000}$.
$792{\cdot}8345 = \frac{792834}{1000} + \frac{5}{10000}$.
$792{\cdot}8345 = \frac{7928345}{10000}$.

EXERCISE LXXV.

☞ Decompose each of the following fractions into four equivalent expressions, as in the example :—

1. 213·5; 40·687. 2. 53·089; 2790·387.
3. 42·5068; 1209·385. 4. 107·98; ·10798.
5. 6305·792; 270·69. 6. 41·372; 809·6274.
7. 61·08; 5·079. 8. 32·765; 10·9721.

257. Every line of figures having a decimal point may be considered the numerator of a fraction, whose denominator is ten raised to the power indicated by the number of figures to the right of the decimal point.

Example.—$\cdot 172 = \frac{172}{10^3} = \frac{172}{1000}$; $20{\cdot}53 = \frac{2053}{10^2}$ or $20\frac{53}{100}$.

$4{\cdot}7296 = 4\frac{7296}{10^4}$ or $\frac{47296}{10000}$; $\cdot 0083 = \frac{83}{10^4} = \frac{83}{10000}$.

258. TO REDUCE A DECIMAL TO THE FORM OF A VULGAR FRACTION.

RULE.

Place the whole of the figures as the numerator of the fraction, omitting the decimal point, and place as the denominator the figure 1, followed by as many ciphers as there are figures to the right of the decimal point.

EXERCISE LXXVI.

☞ Reduce the following decimals into equivalent vulgar fractions.

(When whole numbers occur represent the decimals in two ways. (a). As mixed numbers; (b). As improper fractions.)

1. 235·79; ·0186; 5·072.
2. 8·07; ·123; 1·23.
3. 40·327; 5·69; 247·85.
4. ·005; 6·078; 17·28.
5. 27296·8; 807·324; ·307.
6. 7·129; 18·736; 4·7293.
7. 8·072; 370·296; 10·72.
8. ·0001; 100·7; 308·6.

259. Vulgar fractions cannot always be converted into decimals which are exactly equivalent: but we may always obtain accuracy as nearly as we desire in the decimal form. There are two methods of effecting this change.

260. *I. Method of equal multiplications and divisions.*

Example I.—To reduce $\frac{3}{4}$ to a decimal form is to find a fraction which shall equal $\frac{3}{4}$ and yet shall have 10 or some power of 10 for its denominator. By (194) we may multiply both numbers by 100; then $\frac{3}{4} = \frac{300}{400}$. Divide both by 4; then $\frac{300 \div 4}{400 \div 4} = \frac{75}{100} = \cdot 75$, which is the decimal equivalent to $\frac{3}{4}$.

Example II.—Convert $\frac{5}{7}$ into a decimal form. On multiplying numerator and denominator by 10,000 the fraction becomes $\frac{50000}{70000}$; dividing both terms by 7, we have this result— $7\big|\frac{50000}{70000} = \frac{7142\frac{6}{7}}{10000}$; neglecting the remainder, we have here a fraction $\frac{7142}{10000}$ in a decimal form, and expressible thus ·7142. It is not exactly equal to $\frac{5}{7}$, but it does not differ from it by so much as $\frac{1}{10000}$, and it is evident that by extending the same process we arrive at a still closer approximation to its value. Thus—

$\frac{5}{7} = \frac{5000000}{7000000}$; dividing both by 7 | $\frac{5000000}{7000000} = \frac{714285\frac{5}{7}}{1000000} = \cdot714285.$
Here the decimal expression ·714285, though still not exactly equivalent to $\frac{5}{7}$, does not differ from it by so much as a millionth of a unit, and may therefore be considered practically as its equivalent. There is no limit to the extent to which this process may be carried, and therefore the error may be reduced to as small an amount as we please.

261. *II. Method of reduction.* From (103) it will be seen that the division of all quantities is effected by reducing the remainders step by step into equivalent numbers of a lower name; the same method is applicable here.

Example.—Let it be required to reduce $\frac{5}{19}$ to a decimal fraction, *i.e.*, to divide 5 by 19 in such a way that the answer shall appear in the form of tenths, hundredths, thousandths, &c. For this purpose we will reduce these 5 whole numbers into the required parts.

```
I.  19)5(0 whole numbers
       10
    19)50(2 tenths
       38
       12
       10
    19)120(6 hundredths
       114
         6
        10
    19)60(3 thousandths
       57
        3
       10
    19)30(1 ten thousandth
       19
       11
       10
    19)110(5 hundred thousandths
        95
        15
        10
    19)150(7 millionths
       133
        17
```

II. The contracted process.

```
19)50(263157
   120
     60
      30
     110
      150
       17
```

Here the 19th part of 5 is observed to give no answer in integers, so we reduce the 5 into tenths, and find that the 19th part of 50 tenths gives 2 tenths and 12 tenths over. These 12 tenths are reduced into 120 hundredths, and the 19th part of them is 6 hundredths, with a remainder 6 hundredths. These 6 hundredths are again reduced into the next lower name, and divided by 19; and all the remainders will be seen to have been treated exactly as in Compound Division. The answer, carried as far as millionths, appears to be, $\frac{2}{10} + \frac{6}{100} + \frac{3}{1000} + \frac{1}{10000} + \frac{5}{100000} + \frac{7}{1000000} = \frac{263157}{1000000} = \cdot263157$. Now as reduction into the next power of 10 may simply be effected by adding a cipher, it is evident that we have employed more figures in this example than are necessary. The contracted process gives all the figures which are necessary (102) to obtain the result, and it is evident that we may go on adding ciphers to the remainder, or reducing them to lower names, as long as we please, every step giving us a nearer decimal approximation to the value of $\frac{5}{19}$.

262. TO REDUCE VULGAR FRACTIONS TO A DECIMAL FORM.

RULE.

Divide the numerator by the denominator. If any quotient arises it is an integer number; but if not, or if there be a remainder, add a cipher, and continue the division until there is no remainder.

263. *Observation.*—The quotient obtained after adding one cipher stands in the place of tenths (*i. e.*, next after the decimal point); the quotient obtained after adding two ciphers stands in the second place; after adding 3 ciphers, in the third place; 4 ciphers, in the fourth place, &c.

EXERCISE LXXVII.

☞ Reduce the following fractional expressions to decimals true to the fifth place :—

1. $\frac{4}{7}$; $\frac{519}{284}$; $\frac{81}{29}$. 2. $\frac{3478}{5196}$; $\frac{509}{625}$ $\frac{8}{9}$.

3. $\frac{7}{12}$; $\frac{54}{75}$; $\frac{68}{34}$. 4. $\frac{59}{64}$; $\frac{21}{125}$; $\frac{16}{18}$.

5. $\frac{17}{16}$; $\frac{27}{200}$; $\frac{34}{215}$. 6. $\frac{24}{150}$; $\frac{276}{345}$; $\frac{18}{50}$.

7. $2\frac{7}{8}$; $13\frac{2}{5}$; $122\frac{7}{9}$. 8. $\frac{9}{17}$; $\frac{282}{546}$; $\frac{188}{1794}$.

9. $\frac{180}{576}$; $12\frac{3}{4}$; $2\frac{4}{7}$. 10. $\frac{2}{7}$ of $\frac{5}{8}$; $\frac{4}{5}$ of $\frac{18}{19}$.

11. $\dfrac{\frac{2}{7} \times \frac{5}{17}}{\frac{3}{4} + \frac{2}{3}}$; $\dfrac{18\frac{2}{7} - 3\frac{5}{8}}{16\frac{5}{9}}$. 12. $\dfrac{\frac{4}{5} \text{ of } 7\frac{1}{2}}{\frac{3}{4} \text{ of } 19\frac{2}{5}}$; $\dfrac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{\frac{1}{2} \text{ of } \frac{1}{3} \text{ of } \frac{1}{4}}$.

RECURRING OR CIRCULATING DECIMALS.

264. In many cases, however far we carry the answer, there is still a remainder; that is to say, it is not possible to represent the given fraction exactly in a decimal form. Sometimes it may be seen at once after the operation is begun, that the same remainder occurs a second time, and that consequently the same set of figures will recur in the quotient. Thus—

$$\frac{1}{3} = \frac{10000 \div 3}{30000 \div 3} = \frac{3333}{10000} = \cdot 3333\cdot$$

For as the same remainder occurs every time, the same quotient will recur *ad infinitum*. This is called a RECURRING OR INTERMINATE DECIMAL.

Sometimes also a series of different figures recurs in the same order. On reducing $\frac{2}{7}$ to a decimal we find the following result :—

$$\begin{array}{l} 7)\underline{20\qquad\qquad\qquad} \\ \cdot 285714\ 285714\cdot\ \cdot\ \cdot \end{array}$$

Here, after we have passed the sixth figure of the quotient, we have the remainder 2; but as this is the figure with which we started, we shall of course, the divisor remaining the same, obtain the same set of figures in the quotient again. The decimal ·285714 is therefore called a REPEATING OR CIRCULATING DECIMAL.

When, as in these cases, the same figures recur from the beginning, the expressions are called *Pure Circulating Decimals*. But the fraction $\frac{7}{12}$ for example, is found to equal ·58333, &c. The first two figures, 58, do not repeat, but the 3 does. Such an expression is called a *Mixed Circulating Decimal*.

The figures which are repeated are called the "repetend." When only one figure recurs it is called a "simple repetend;" when more than one, they form what is called a "compound repetend." Thus in the circulating decimal ·333, &c., 3 is a simple repetend. In ·962962962, &c., 962 is a compound repetend.

It is usual to indicate a circulating decimal by placing a point over the first and last of the recurring digits. Thus—

$$\begin{array}{lcl} \cdot\dot{6} & = & \cdot 666\cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\dot{2}\dot{7} & = & \cdot 272727,\ \&c. \\ \cdot 58\dot{3} & = & \cdot 58333,\ \&c. \\ \cdot\dot{2}9\dot{6} & = & \cdot 296296296,\ \&c. \end{array}$$

265. *Every vulgar fraction whose value cannot be exactly expressed decimally, will take the form of a circulating decimal.*

For as every remainder must be less than the divisor, the number of remainders which can occur is limited; and because a cipher is added to every remainder, when any dividend occurs a second time, the same set of dividends and quotients must appear as before.

EXERCISE LXXVIII.

☞ Reduce the following fractions to circulating decimals :—

1. $\frac{7}{11}$; $\frac{5}{9}$; $\frac{23}{24}$. 2. $\frac{4}{7}$; $\frac{7}{12}$; $\frac{8}{19}$. 3. $\frac{5}{13}$; $\frac{6}{17}$; $\frac{4}{15}$.
4. $\frac{17}{23}$; $\frac{12}{19}$; $\frac{6}{27}$. 5. $\frac{8}{17}$; $\frac{5}{13}$; $\frac{3}{7}$. 6. $\frac{12}{13}$; $\frac{2}{3}$; $\frac{8}{9}$.
7. $\frac{3}{7}$; $\frac{3}{35}$; $\frac{9}{11}$. 8. $\frac{15}{33}$; $\frac{7}{9}$; $\frac{10}{55}$. 9. $\frac{11}{12}$; $\frac{6}{7}$; $\frac{15}{17}$.

266. It must be observed that in every repeating decimal, the whole value of the quantity represented is not expressed; for as the series is infinite we may carry it as far as we will, and yet some part of the fraction will remain unwritten. These parts may be very small, yet they are truly parts of the fraction, and in the decimal system we are obliged to neglect them. The following is an easy method for finding a vulgar fraction, which represents the exact value of a pure circulating decimal; and it will be seen that, whereas by prolonging the decimal line, we continually approach, but never reach, the true expression of its value, it is always possible to give an accurate expression in the form of a vulgar fraction.

Example I.—Let S represent the sum of the series ·4444· · ·

(a). Then (255) 10 S = 4·4444· · · · · ·
(b). But S = ·4444· · · · · ·

Subtracting (b) from (a) 9 S = 4 therefore S = $\frac{4}{9}$.

Here, as it is manifest that the indefinite expression ·4444· · · · means the same in both lines, the one may be subtracted from the other, and will leave no remainder. But as 10 S — S = 9 S, it follows that 4 equals 9 times the required fraction, which is therefore $\frac{4}{9}$.

Example II.—Let the circulating decimal be ·252525; then, because there are two figures in the repetend it will be convenient to multiply the sum by 100. Then as before, let S = $\cdot\dot{2}\dot{5}$—

Then 100 S = 25·252525, &c.
Subtract S = ·252525, &c.

99 S = 25. Therefore S = $\frac{25}{99}$.

Example III.—Let the recurring decimal be ·142857; and as its denominator is a million, it will be convenient to multiply by that number. Here as before, let S = ·$\dot{1}4285\dot{7}$—

Then 1000000 S = 142857·$\dot{1}4285\dot{7}$, &c.
Subtract S = ·$\dot{1}4285\dot{7}$, &c.

999999 S = 142857. And S = $\frac{142857}{999999}$.

267. When Mixed Circulating Decimals occur they should be treated as follows :—

I. Reduce ·583$\dot{3}$3 to a vulgar fraction.

Multiply by 1000 Then 1000 S = 583·$\dot{3}$3 (a)
Multiply by 100 100 S = 58·33 (b)

Subtracting (b) from (a) 900 S = 525 ∴ S = $\frac{525}{900}$.

II. Reduce ·94$\dot{7}2\dot{4}$ to a vulgar fraction.

Multiply by 100000 Then 100000 S = 94724·$\dot{7}2\dot{4}$
Multiply by 100 Then 100 S = 94·724

Subtract Then 99900 S = 94630
∴ S = $\frac{94630}{99900}$.

In each of these cases we first multiply by a power of ten, high enough to make an integer number of the recurring and the non-recurring parts; we then multiply the expression by a power of 10, high enough only to make a whole number of the non-recurring part; on subtracting this from the other, the circulating decimal disappears, and the whole number which results has for its denominator the difference between the greater power of ten and the less.

268. TO REDUCE A CIRCULATING DECIMAL TO A VULGAR FRACTION.

If only the same figures recur, make the repetend the numerator of a fraction, and place underneath it as many nines as there are digits in the repetend.

But if it be a mixed circulating decimal, subtract the digits which do not recur from the whole expression as far as the end of the first repetend; take this difference as the numerator, and for the denominator place as many nines as there are digits in the recurring part, followed by as many ciphers as there are digits in the part which does not recur.

EXERCISE LXXIX.

☞ Find vulgar fractions equivalent to the following expressions:—

1. $\cdot\dot{4}\dot{7}$; $\cdot\dot{5}$; $\cdot0\dot{5}$. 2. $\cdot\dot{3}2\dot{7}$; $\cdot\dot{9}523\dot{8}$; $\cdot\dot{7}1428\dot{5}$.
3. $\cdot00\dot{1}8\dot{5}$; $\cdot3\dot{1}\dot{3}2$; $1\cdot70\dot{1}\dot{6}$. 4. $\cdot28\dot{1}7\dot{2}$; $3\cdot4\dot{1}\dot{5}$; $\cdot70\dot{2}3\dot{8}$.
5. $6\cdot\dot{0}3\dot{8}$; $\cdot71\dot{2}\dot{5}$; $\cdot80\dot{5}6\dot{3}$. 6. $\cdot13\dot{5}8\dot{7}$; $\cdot\dot{7}1\dot{9}$; $6\cdot0\dot{2}\dot{7}$.
7. $1\cdot27\dot{3}1\dot{6}$; $45\cdot50\dot{8}$; $\cdot\dot{8}2\dot{7}$. 8. $\cdot\dot{7}0\dot{3}2$; $6\cdot4\dot{1}$; $7\cdot0\dot{4}\dot{3}$.
9. $27\cdot0\dot{5}\dot{6}$; $3\cdot00\dot{7}\dot{5}$; $4\cdot\dot{1}2\dot{7}$. 10. $11\cdot\dot{3}7\dot{2}$; $4\cdot16\dot{8}$; $\cdot5\dot{0}93\dot{2}$.
11. What is the difference between $\cdot\dot{0}\dot{7}$ and $\cdot0\dot{7}$?

269. *No vulgar fraction can be precisely expressed as a decimal if its denominator can be resolved into any other prime factors than 2 and 5.*

Demonstrative Example.—The fraction $\frac{7}{9}$ cannot be reduced lower because its terms are prime, but (196) any fraction can be altered into the form of one of a higher name, provided that the new denominator is a multiple of the former. If either 10, or 100, or 1000, or any power of 10 be also a multiple of 9, the fraction $\frac{7}{9}$ can be reduced to its exact equivalent in the decimal system, but if not, the fraction must take the form of a repeating decimal.

The question to solve in this case is, "Can any multiple whatever of the number 9 be found equal to any power of ten?" If it can, then 9 must be a measure of either 10 or some power of ten. But it may be inferred from (168) that if a number measure another that other must contain all the prime factors of the first. Now the only prime factors of ten and of all the powers of ten are 2 and 5, and neither of these is a factor of 9; wherefore no multiple of 9 can ever equal a number whose prime factors are 2 and 5. In the same manner it might be proved, that whenever the denominator of a vulgar fraction, expressed in its lowest form, is resolved into any other prime factors than 2 or 5, the fraction will be interminate.

270. TO DETERMINE WHETHER A VULGAR FRACTION CAN BE EXPRESSED DECIMALLY.

RULE.

Reduce the vulgar fraction to its lowest terms (198). Find the prime factors of the denominator (159). If they are any others than 2 or 5 the fraction will take the form of a repeating decimal.

Example.– In the fraction $\frac{7}{32}$ the denominator $32 = 2 \times 2 \times 2 \times 2 \times 2$. and has no other prime factor than 2: the fraction may therefore be expressed decimally. But in $\frac{8}{15}$, because $15 = 3 \times 5$, and no power of 10 has 3 for a prime factor, the fraction is interminate.

EXERCISE LXXX.

☞ State which of the fractions in Exercise LVIII. are capable of being expressed as decimals, and give the reason.

271. When decimal fractions are extended to any great length, it becomes very cumbersome to work with them. The figures which stand in the fifth, sixth, and later places, represent exceedingly small values, are only needed in calculations of great nicety, and are often neglected. Whenever this is done it is necessary to look at the first figure which is neglected, and to attend to the following considerations.

Suppose the fraction 17·168327 is given, and we only wish to deal with it as far as the second place of decimals; neglecting the four figures to the right we call it 17·16. Now the 8, which is the first of the digits cut off, means $\frac{8}{1000}$, and if $\frac{2}{1000}$ were added to it would become $\frac{10}{1000}$ or $\frac{1}{100}$, which is a number suitable to be carried into the next place to the left. When therefore I write the fraction as 17·16 the expression differs by $\frac{8}{1000}$ from the truth, but if I had written 17·17 it would only have differed by $\frac{2}{1000}$ from the truth. In this case we make a greater error by omitting it altogether than by adding one to the place on its left. Hence 17·17 is nearer to 17·168 than 17·16 is. But as the next figure to the right is 3, we should have to add $\frac{7}{10000}$ to it to make the fraction 17·169, whereas by leaving it out, and simply writing 17·168, we only depart $\frac{3}{10000}$ from the truth.

If 5 were the number, the error of omitting it would be the same as that of counting it as 1 in the next place to the left.

Thus ·237 true to two places = ·24, because 37 is nearer to 40 than to 30.

But 7·2835 true to two places = 7·28, because 835 is nearer to 800 than to 900.

In like manner ·718 true to two places = ·72.
·657 .... = ·66.
But ·314 .... = ·31.

272. TO FIND THE NEAREST EQUIVALENT TO ANY FRACTION AS FAR AS TO ANY GIVEN PLACE OF DECIMALS.

RULE.

If the first of the neglected figures be 5 or more than 5, add 1 to the last of the digits which are retained. But if the first of the rejected figures be less than 5, the former figures may be set down without them.

EXERCISE LXXXI.

☞ (*a*) Find the nearest equivalent to the following decimals, and also to those in Exercise LXXVI., as far as two decimal places.

1. 15·687
2. 7·2358
3. 9·1742
4. 2·6189
5. ·712
6. ·843
7. ·296
8. ·4312
9. ·2145
10. ·019
11. ·007
12. 5·3684

(*b*) Make a table similar to the following, comprising all vulgar fractions in the series from $\frac{1}{13}$ to $\frac{19}{20}$, true to four places of decimals.

$\frac{1}{2} = \cdot 5$
$\frac{1}{3} = \cdot 33\dot{3}$
$\frac{2}{3} = \cdot 66\dot{6}$
$\frac{1}{4} = \cdot 25$
$\frac{3}{4} = \cdot 75$
$\frac{1}{5} = \cdot 2$
$\frac{2}{5} = \cdot 4$
$\frac{3}{5} = \cdot 6$
$\frac{4}{5} = \cdot 8$
$\frac{1}{6} = \cdot 166\dot{6}$
$\frac{5}{6} = \cdot 833\dot{3}$
$\frac{1}{7} = \cdot 1428 \cdot\cdot$
$\frac{2}{7} = \cdot 2857 \cdot\cdot$
$\frac{3}{7} = \cdot 4285 \cdot\cdot$
$\frac{4}{7} = \cdot 5714 \cdot\cdot$

$\frac{5}{7} = \cdot 7142 \cdot\cdot$
$\frac{6}{7} = \cdot 8571 \cdot\cdot$
$\frac{1}{8} = \cdot 125$
$\frac{3}{8} = \cdot 375$
$\frac{5}{8} = \cdot 625$
$\frac{7}{8} = \cdot 875$
$\frac{1}{9} = \cdot 11\dot{1}$
$\frac{2}{9} = \cdot 22\dot{2}$
$\frac{4}{9} = \cdot 44\dot{4}$
$\frac{5}{9} = \cdot 55\dot{5}$
$\frac{7}{9} = \cdot 77\dot{7}$
$\frac{8}{9} = \cdot 88\dot{8}$
$\frac{1}{10} = \cdot 1$
$\frac{3}{10} = \cdot 3$
$\frac{7}{10} = \cdot 7$

$\frac{9}{10} = \cdot 9$
$\frac{1}{11} = \cdot 09\dot{0}\dot{9}$
$\frac{2}{11} = \cdot 18\dot{1}\dot{8}$
$\frac{3}{11} = \cdot 27\dot{2}\dot{7}$
$\frac{4}{11} = \cdot 36\dot{3}\dot{6}$
$\frac{5}{11} = \cdot 45\dot{4}\dot{5}$
$\frac{6}{11} = \cdot 54\dot{5}\dot{4}$
$\frac{7}{11} = \cdot 63\dot{6}\dot{3}$
$\frac{8}{11} = \cdot 72\dot{7}\dot{2}$
$\frac{9}{11} = \cdot 81\dot{8}\dot{1}$
$\frac{10}{11} = \cdot 90\dot{9}\dot{0}$
$\frac{1}{12} = \cdot 083\dot{3}$
$\frac{5}{12} = \cdot 416\dot{6}$
$\frac{7}{12} = \cdot 583\dot{3}$
$\frac{11}{12} = \cdot 916\dot{6}$

Note.—Equivalent decimals are to befound for those vulgar fractions only that are of a *different* value from any which have preceded them; *e.g.*, because $\frac{5}{15} = \frac{1}{3}$, the decimal answering to which has been already obtained, its equivalent need not be found again.

SECTION IX.—ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF DECIMAL FRACTIONS.

ADDITION.

273. From (199) it appears that when fractions are reduced to a common denominator the numerators alone may be added or subtracted to give the sum or difference of any two fractions. But in Decimals the numerators only of the fractions are expressed, and the unexpressed denominators are easily made common by the mere arrangement of the figures in a certain order.

Example I.—$7{\cdot}2916 + 80{\cdot}25 + 2423{\cdot}41 + 8071{\cdot}2 + 7056{\cdot}8103$.

$$\begin{array}{rlll}
7{\cdot}2916 & = & & \frac{72916}{10000} \\
80{\cdot}2500 & = \frac{8025}{100} & = & \frac{802500}{10000} \\
2423{\cdot}4100 & = \frac{242341}{100} & = & \frac{24234100}{10000} \\
8071{\cdot}2000 & = \frac{80712}{10} & = & \frac{80712000}{10000} \\
7056{\cdot}8103 & = & & \frac{70568103}{10000} \\
\hline
17638{\cdot}9619 & = & & \frac{176389619}{10000}
\end{array}$$

Here ciphers have been added in order to reduce all the fractions to the common denominator 10000; and it is evident that, neglecting the consideration of the decimal point altogether, every line of figures may be regarded as an integer number, the numerator of a fraction whose denominator is 10000. On adding these numerators together, as whole numbers, the sum is found to be 176389619; and as the common denominator is 10000, or the fourth power of 10, the point must be placed four figures from the right, and the answer is 17638·9619.

The ciphers which have been introduced are not necessary, as the value of each figure is sufficiently known by its position.

274. *Example II.*—Add together ·0018, 7·96, 413·587, 21·409.

$$\begin{array}{r}
{\cdot}0018 \\
7{\cdot}96 \\
413{\cdot}587 \\
21{\cdot}409 \\
\hline
442{\cdot}9578
\end{array}$$

Here the figure standing furthest to the right is 8, and means $\frac{8}{10000}$; there are no other figures of that value, so we set down 8 in the fourth place from the unit. In the next column are 9, 7, and 1, or 17; these are $\frac{17}{1000}$, but $\frac{17}{1000} = \frac{10}{1000} + \frac{7}{1000} = \frac{1}{100} + \frac{7}{1000}$, so the 7 are put in the thousandths place, and $\frac{10}{1000}$, or $\frac{1}{100}$, are carried into the hundredths place as 1. This 1 added to 8 and 6 makes 15, and these are 15 hundredths; but $\frac{15}{100} = \frac{1}{10} + \frac{5}{100}$, we therefore set down the 5 as hundredths, and carry 1 to the tenths. $\frac{1}{10} + \frac{4}{10} + \frac{6}{10} + \frac{8}{10} = \frac{19}{10}$, but $\frac{19}{10} = 1 + \frac{9}{10}$, we therefore set down the 9 among the tenths and carry the one to the other side of the decimal point as a whole number. The rest is the ordinary Addition of Integers.

275. TO ADD DECIMAL FRACTIONS.

RULE.

Arrange the figures so that the decimal points in all the lines shall fall in one vertical column. Add up as in Simple Addition, placing a decimal point in the answer exactly underneath the other points.

EXERCISE LXXXII.

1. 279·806 + 304·72 + ·008 + ·596 + 2·037.
2. 108·62 + ·001 + ·1007 + 3·8 + 173·6 + 17·365.
3. 58·72 + 96·057 + 41·2874 + 3·027 + 1865·07.
4. 71·9683 + 2·17 + 4·621 + ·008 + 72·0963 + ·04.
5. 632·1874 + 4·017 + 163·5 + 8·047 + 4·198 + 3·74.
6. 71·827 + 3·142 + 6·1547 + 103·03 + 51·0008.
7. 820·95 + 70·03 + ·008 + 10·72 + 13·5678.
8. 927·416 + 8·274 + 372·6 + 62·07938 + ·507462.
9. 103·72 + 11·7 + 61·187 + 3·0972 + 2·073 + 864·145.
10. 27·07 + 83 6 + 409·6 + 3·1725 + 8·627 + ·0072.
11. 2·124 + 8·327 + 65·47 + ·2198 + 327·4 + 862·1.
12. 57·213 + 8627·9 + 4138·7 + 65·41 + ·00728 + 1·05.
13. Add together 17 thousandths, 2 tenths, and 47 millionths.
14. Add together 238 tenths, 453 thousandths, 6134 millionths, and 18 ten thousandths.
15. Find the sum of 264 hundredths, 18 tenths, 34 millionths, and 62,584 hundred thousandths.
16. Find the value to 7 decimal places of $\cdot 728\dot{3} + \cdot 14\dot{2}85\dot{7} + 7\cdot 3\dot{1}\dot{6} + 2\cdot 0\dot{6} + 5\cdot 783$.
17. Add together 257·328, $5\cdot 09\dot{8}$, $16\cdot 23\dot{5}\dot{7}$, $14\cdot 0\dot{4}\dot{3}$, $2\cdot\dot{8}6\dot{2}$, 5·089721, $3\cdot 1\dot{6}2$, and $851\cdot\dot{2}\dot{7}$, making the answer true to the eighth decimal place.

SUBTRACTION.

276. This Rule, like Addition, is always a simple process when the fractions have a common denominator. Thus, if it be required to take 7·314 from 25·06, we have (257) two vulgar fractions, $\frac{7314}{1000}$ and $\frac{2506}{100}$, of which the latter, $\frac{2506}{100}$, can be brought to the same name as the former by adding a cipher to the numerator and denominator. It then becomes $\frac{25060}{1000}$. We have now to find the difference between $\frac{7314}{1000}$ and $\frac{25060}{1000}$. Taking (210) the difference of the numerators only we find--

$$\frac{25060}{1000} - \frac{7314}{1000} = \frac{17746}{1000} = 17{\cdot}746.$$

$$\begin{array}{r} 25{\cdot}060 \\ \underline{7{\cdot}314} \\ 17{\cdot}746 \end{array}$$

Or, 25·060 7·314 17·746 Here we simply place the figures of like values under one another, assume a cipher in the upper place on the right, and proceed as in Simple Subtraction.

277. *Observation.*—On examining sums of this kind more closely we shall find that what was said of Simple Subtraction is equally true here. We do not actually subtract the required quantity from the other, but in nearly all cases we add something to both ; and we take the subtrahend + this added number, from the minuend + the same number. Thus in the sum, subtract 20·758 from 301·2439,

	hundreds.	tens.	units.	tenths.	hundredths.	thousandths.	ten thousths.
301·2439 =	3	10	1	12	14	13	9
20·758	1	2	1	8	6	8	
280·4859	2	8	0	·4	8	5	9

instead of 3, in the thousandths place of the minuend, we have taken 13 ; but we have also turned the 5, of the hundredths place of the subtrahend, into 6. Thus to the upper line we have added $\frac{10}{1000}$, and to the lower $\frac{1}{100}$. But these are equal, therefore, by (44) they do not affect the answer. So also we have added $\frac{10}{100}$ to the 4 in the upper line, and $\frac{1}{10}$, which is the same as $\frac{10}{100}$, to the 7 of the lower line. $\frac{10}{10}$ have been added to the tenths of the upper line, and 1, which is equal to $\frac{10}{10}$, has been added in the units place of the lower line. In like manner 10 tens have been added to the minuend, and 1 hundred to the subtrahend. Thus equal additions have been made to both lines, and the work really effected has been,

$$\begin{array}{ll} \text{From} & 301{\cdot}2439 + 10 \text{ tens} + \frac{10}{10} + \frac{10}{100} + \frac{10}{1000} \\ \text{Take} & \underline{20{\cdot}758 + 1 \text{ hundred} + 1 + \frac{1}{10} + \frac{1}{100}} \\ & 280{\cdot}4859 \end{array}$$

278. To SUBTRACT ONE DECIMAL FRACTION FROM A GREATER.

RULE.

Arrange the numbers so that the points are in a vertical line, and that figures of the same value are in corresponding places. Subtract as in Simple Subtraction, and place a point in the answer underneath the other two.

EXERCISE LXXXIII.

☞ Work the following sums in Subtraction:—

1. 709·63 − ·8514; 234·057 − 18·5.
2. 72·065 − 19·7234; 81·963 − 1·7.
3. 2107·5462 − 17·19382; 18 − ·18.
4. 150·7 − 1·507; 216·9 − ·83472.
5. 1201·6 − 43·598; 47·106 − 8·271.
6. 1 − ·001; 827·43 − 97·6387.
7. 2123·5 − ·0078; 62·97 − ·63845.
8. 2172·81 − 31·629; 41·78 − ·29643.
9. 872·1 − (·8721 + ·008); 378·69 − (20·07286 + 81·6).
10. 34·72 + ·1862 − 6·82; 27·8 + 619·7 − 327·9.
11. (5·028 + ·0073) − (6·704 − 2·38).
12. 3·7246 + 4·1 + ·097 − 7·42; $7{\cdot}1\dot{8} + 62{\cdot}\dot{3} - {\cdot}\dot{5}$.
13. $4{\cdot}46 - {\cdot}1\dot{9}\dot{7}$; 80·23 − 7·6453.

14. What is the difference between the sum of 33 millionths and 17 thousandths, and the sum of 53 hundredths and 274 tenths?

15. What is the difference between $\frac{3}{8}$ and 1·0084?

16. How much greater is the difference between $2\frac{1}{8}$ and 1·256 than the sum of ·05684 and ·556?

17. A person had ·825 of a ship left after he had sold a share amounting to $\frac{5}{40}$; what part of the ship was his at first?

18. Say by how much the product of $2\frac{5}{7}$ and $\frac{3}{4}$ exceeds the difference between $3\frac{2}{5}$ and 1·98046

19. Which is the greater, and by how much, $\frac{3}{8}$ of $\frac{5}{8}$ of £50, or £8·642 + £18·0564?

20. The franc weighs 77·17 grains, of which 69·453 are pure silver; what is the weight of the alloy?

21. What is the difference between the mean annual temperature of Dublin and Petersburg, the former being 49·05°, the latter 39·61°?

MULTIPLICATION.

279. From (214) it appears that to multiply two fractions together we multiply the numerators together for the numerator of the product, and the denominators together for the new denominator. This principle applies also to decimals. Now it is easy to multiply the numerators, which in Decimals are always expressed in figures; but as the denominators are not expressed, but only indicated by the pointing, it requires some care to know what the denominator of the product is, and how its value can be accurately marked.

Let it be required to find the product of ·005 and 1·27. This (must be equivalent to $\frac{5}{1000} \times \frac{127}{100}$. But by (219) the product of these two fractions is $\frac{635}{100000}$, which, expressed decimally, is ·00635. Here, because 5 is one numerator and 127 the other, $5 \times 127 =$ the numerator of the product. But as in ·005 the 5 stands in the third place from the unit, it represents 5 divided by the third power of 10. Again, because 1·27 means 127 divided by the second power of 10, the two denominators are 10^3 and 10^2.

$$\text{But } 10^3 = 1000, \text{ and } 10^2 = 100;$$
$$\text{And } 1000 \times 100 = 100000, \text{ or } 10^3 \times 10^2 = 10^5.$$

In like manner, the product of any powers of 10 is represented by adding the numbers which are the exponents of those powers. Thus $10^2 \times 10^7 = 10^{2+7} = 10^9$. So also $10^4 \times 10^3 = 10^{4+3} + 10^7$. Now, because the denominator of every decimal fraction is always that power of 10 which is indicated by the number of figures to the right of the decimal point, it follows, that by adding together the number of places on the right of the point in both factors, we learn how many places should be at the right of the point in the product. The following examples will illustrate this :—

1. $\cdot 5 \times \cdot 7 = \frac{5}{10} \times \frac{7}{10} = \frac{5 \times 7}{10 \times 10} = \frac{35}{100} = \cdot 35.$
2. $\cdot 004 \times 1\cdot 7 = \frac{4}{10^3} \times \frac{17}{10} = \frac{68}{10^4} = \cdot 0068.$
3. $5 \times \cdot 00003 = 5 \times \frac{3}{10^5} = \frac{15}{10^5} = \cdot 00015.$
4. Multiply 2780·961 by 1·32.

$$\begin{array}{r l}
2780\cdot961 & = 2780961 \div 10^3 \\
1\cdot32 & = 132 \div 10^2 \\
\hline
5561922 & \\
8342883 & \\
2780961 & \\
\hline
3670\cdot86852 & = \frac{2780961 \times 132}{10^5}
\end{array}$$

280. TO MULTIPLY DECIMAL FRACTIONS.

RULE.

Multiply the numbers as integers. Mark off as many places to the right of the decimal point in the product as there are in all the factors put together.

EXERCISE LXXXIV.

☞ Find the product of the following decimal expressions :—

1. 27·98 × 63·5 ; 20·5 × 318·62.

2. 3·5 × 4·7 × 18·05 ; 16 × 27·1 × ·817.

3. 712·4 × 81·67 × 21 ; 2·03 × 203 × ·203.

4. 17·186 × ·5198 ; ·007 × 31·5 × ·617.

5. 61·58 × 317·2 ; 31·87 × 61·98 × 214·7.

6. 713·72 × 81·076 × 2·03 ; 8·145 × 614·7 × 30·03.

7. (2·7 + 31·85) × (3·16 − ·316) ; (4·198 − 1·7096) × 31·278.

8. (12·7 × 3·16 × 2·1) + (27·3 × 10·8) ; (61·95 + 87·2) − (21·3 × 6·19).

9. (71·42 × 3·164 × 21·008) − 17·45 ; ·002 × 2 × 65·7.

10. (107·8 + 6·541 − 31·96) × 1·742 ; 834·72 − (61·35 × 2·007).

11. 2178·5 × 18·74 × 21·72 ; (61·87 + 2·19 − 3·07) × 4·86.

12. Find the product of the sum and difference of 21·8 and ·324.

13. Work the last four sums in Exercise LXIII. decimally.

14. One brother owns ⅞ of a vessel, the whole of which is worth £8,000, while a second has for his portion 72 shares in a railway, worth £75·125 each ; whose property is the more valuable?

15. If in a certain town the number of persons dying every week is 78·4, what number may be assumed to die in the year of 365 days 5 hours 48 minutes 57 seconds?

16. If £96·54 represent the value of an acre of land on a certain estate, what is the worth of the whole, consisting of 1864 acres 3·45 roods?

17. Add the rent of 53·7219 acres of land at £4·12 per acre per annum, to that of 4·05 acres at £3·75 per annum.

18. If the value of gold is 15·1 times that of silver, what weight of silver can be obtained for 7·564 lbs. of gold ?

DIVISION.

281. Whenever one decimal fraction has to be divided by another, if ciphers be added to that which has fewer places of decimals until both have the same number, the decimal points may be omitted, and the dividend and divisor will form the numerator and denominator of a vulgar fraction, which may be reduced to a decimal by the rule given in (262).

282. For by (228) whenever two fractions have a common denominator the one may be divided by the other, by simply dividing the numerator of the dividend by the numerator of the divisor. Therefore, in dividing one decimal expression by another it is only necessary to bring both to the same name, and then the process of division will be the same as the division of integers.

Example I.—Thus, to divide 1·5 by ·003 is to find how many times ·003 or $\frac{3}{1000}$ is contained in 1·5 or $\frac{15}{10}$. But 1·5 or $\frac{15}{10} = \frac{1500}{1000}$. Therefore the sum may take this form, $\frac{1500}{1000} \div \frac{3}{1000}$, and this (228) is equivalent to 1500 ÷ 3, or 500. Wherefore 1·5 ÷ ·003 = 500.

By adding ciphers to 1·5 so as to make it a fraction of the same name as the other, the sum would at once have assumed a form for easy division, 1500 ÷ 3.

Example II.—Divide 27·9 by 16·08. Here, because 27·9 = $\frac{279}{10}$, or $\frac{2790}{100}$, and 16·08 = $\frac{1608}{100}$, the question is—Divide 2790 by 1608, *i. e.*, $\frac{2790}{1608}$. The result is a vulgar fraction, which may be reduced to a decimal by (262), and the answer, though it may prove interminate, can be carried to any degree of accuracy required.

283. The following sums in Division of Decimals take the form of vulgar fractions, which may afterwards be resolved into decimals:—

$70{\cdot}960 \div 2{\cdot}1 = \frac{70960}{1000} \div \frac{2100}{1000} = \frac{70960}{2100}.$
${\cdot}001 \div {\cdot}1 = \frac{1}{1000} \div \frac{100}{1000} = \frac{1}{100}.$
$2970{\cdot}8 \div 1{\cdot}0376 = \frac{29708000}{10000} \div \frac{10376}{10000} = \frac{29708000}{10376}.$
$2{\cdot}0986 \div 1573{\cdot}5 = \frac{20986}{10000} \div \frac{15735000}{10000} = \frac{20986}{15735000}$
${\cdot}029 \div 507{\cdot}356 = \frac{29}{1000} \div \frac{507356}{1000} = \frac{29}{507356}.$
$1{\cdot}817 \div {\cdot}057 = \frac{1817}{1000} \div \frac{57}{1000} = \frac{1817}{57}.$
${\cdot}28 \div 19{\cdot}6534 = \frac{2800}{10000} \div \frac{196534}{10000} = \frac{2800}{196534}.$

Problems in Division of Decimals assume various forms; they usually fall under one of the three following heads:—

284. CASE I.—WHEN THE DIVISOR IS A WHOLE NUMBER.

RULE.

Divide by the whole number, placing the first figure of the quotient in the same decimal place as the figure of the dividend from which it was obtained. Add ciphers, and continue the operation as far as may be necessary.

Example.—Divide 234·729 by 8, and 5·4726 by 12.

$$8\underline{)\ 234{\cdot}729\qquad\qquad}\qquad\qquad 12\underline{)\ 5{\cdot}4726}$$
$$29{\cdot}34112\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\qquad\qquad\qquad {\cdot}45605^{\cdot}$$

In the former of these cases 23 is divided by 8, and a quotient 2 is obtained. But the 23 represents 23 tens, wherefore the eighth part of 23 represents a number of tens, and the 2 must stand in the tens place. In the second case, because the 54 are tenths, the question, what is the 12th part of 54, gives the answer in tenths. The same answers would have been obtained if we had reduced the divisor into a fraction having the same denominator as the dividend, but the process would in this case have been more tedious.

Thus $234{\cdot}729 \div 8 = \frac{234729}{1000} \div \frac{8000}{1000} = \frac{234729}{8000} = 29{\cdot}34112\ \cdot\ \cdot$
And $5{\cdot}4726 \div 12 = \frac{54726}{10000} \div \frac{120000}{10000} = \frac{54726}{120000} = {\cdot}45605.$

EXERCISE LXXXV.

1. 37·58 ÷ 6; 4·096 ÷ 17; 2158·9 ÷ 143.
2. 12·198 ÷ 4; 3·72096 ÷ 12; 41·793 ÷ 271.
3. 105·7096 ÷ 127; 3·01869 ÷ 500; 2108·962 ÷ 423.
4. 5079·638 ÷ 65; 2109·683 ÷ 49; ·0738 ÷ 3.
5. ·139655 ÷ 28; 4172·98 ÷ 37; 1·0076 ÷ 57.
6. ·031712 ÷ 227; 61·087 ÷ 35; 2·0198 ÷ 7.
7. Seven persons gained £374·56, what was each person's share?
8. A field of 15·42 acres, paying a rent of £18·425, was allotted among 16 labourers; what was the size of each man's allotment, and what had he to pay for it yearly?
9. Multiply 798·362 by 15·2, and divide the result by 18.
10. If 29 articles cost 17·35 shillings, what will be the price of 582·3 such articles?

285. CASE II.—WHEN THE DIVISOR HAS A GREATER NUMBER OF DECIMAL PLACES THAN THE DIVIDEND.

RULE.

Add enough ciphers to the dividend to make both have an equal number of places. Then divide, and the quotient will represent whole numbers. If more ciphers have to be added to the dividend, follow the rule given for reducing vulgar fractions to decimals (262).

Example I.—Divide 7·9 by 4·308.

Here 7·9 = $\frac{79}{10}$ = $\frac{7900}{1000}$;
and 4·308 = $\frac{4308}{1000}$.

∴ 7·9 ÷ 4·308 = 7900 ÷ 4308.

4308) 7900 (1·8337
35920
14560
16360
34360
4204
Answer, 1·8337.

Example II.—Divide ·5 by 1·784.

Here ·5 = $\frac{5}{10}$ = $\frac{500}{1000}$;
and 1784 = $\frac{1784}{1000}$.

∴ ·5 ÷ 1·784 = 500 ÷ 1784.

1784) 500·0 (0·28026
14320
4800
12320
1616
Answer, ·28026.

EXERCISE LXXXVI.

1. 204·7 ÷ ·0385 ; 17 ÷ ·66 ; 29 ÷ 5.
2. 3287·45 ÷ 1·638; 2 ÷ ·002 ; 7·4 ÷ 1·38.
3. 71 ÷ ·635 ; 41·472 ÷ 30·5698 ; 301 ÷ 1·7045.
4. 201 ÷ 5473; 1862 ÷ ·54 ; 7001 ÷ ·1007.
5. 712 ÷ ·035; 1·27 ; ÷ ·127 10 ÷ ·10.
6. (37·29 × ·54) ÷ 1·876; (4·7 + 3·09 ÷ (·7 × ·854).
7. (20·9 − 16·437) ÷ (·52 + ·105 − ·03) ; 30·04 ÷ ·4003.
8. (7·9 × 6·8) ÷ (·34 + ·28 + 1·695); 40·093 ÷ (8·1 of 20·574).
9. Divide the product of 2·63 and 179 by that of 1·084 and 21.695.
10. Divide the sum of 18·3 and 21·065 by the difference between 17·09867 and 25·1.
11. In each franc piece there are 69·453 grains of pure silver; how many francs can be made from 25 lbs. of silver?
12. The ancient Roman mile was ·917 of an English mile; how would the length of the diameter of the earth be expressed in Roman miles, it being 7,926 English miles?

286. CASE III.—WHEN THE DIVIDEND HAS A GREATER NUMBER OF DECIMAL PLACES THAN THE DIVISOR.

RULE.

Divide as in whole numbers. Mark off in the answer as many decimal places as the dividend contains more than the divisor. Add ciphers, and carry the answer to any place of decimals required.

It would add to the trouble of calculation in this case if we were to bring both divisor and dividend to a common denominator. Thus if it be required to divide 7·9384 by ·8: this means (282), divide $\frac{79384}{10000}$ by $\frac{8000}{10000}$, or divide 79384 by 8000. If this were done the answer would be in whole numbers. But if instead of dividing by 8000 we divide by 8, and then when the answer is obtained we remember that it is 1000 times too much, and point off three figures accordingly, the sum will be worked more easily.

Example I.—Divide 27·3476 by ·15.

15)27·3476000
182·31733 . . . .

Because the dividend has 7 places of decimals and the divisor only 2, the answer has 5, or 7 - 2.

Example II.—Divide 34·79628 by 2·5.

25)34·79628 (13·9185
97
229
46
212
128
3

Answer, 13·9185: because as the dividend has 5 places and the divisor 1, the quotient must have 4.

287. *Observation.*—The reason of this rule will be further evident from the following considerations:—

Division is the inverse process of Multiplication. It has been shown in (279) that the product of any two decimal fractions must have as many decimal places as there are in both of the factors. But in Division the divisor is one factor and the quotient the other, their product being the dividend (129). Hence, as the number of decimal places in the dividend equals the sum of the number of decimal places in divisor and quotient, it follows that the number of decimal places in the quotient is found by taking the difference between the number in the dividend and the number in the divisor.

EXERCISE LXXXVII.

1. ·0001 ÷ ·01; 7·9285 ÷ ·45; 2·01 ÷ 1·7.
2. 51·78 ÷ 1·1; 3·0724 ÷ 178·36; 40·735 ÷ 185·5.
3. 720·397 ÷ 21·8; 615·821 ÷ 2·4; 3·271 ÷ ·09.
4. 1·2748 ÷ ·53; 61·423 ÷ ·01; 39·7286 ÷ 5·7.
5. 11·723 ÷ 6; 20·1783 ÷ 31·562; 4·0198 ÷ 27·3.
6. 14·05 ÷ 14·5; 3·7298 ÷ 1·27; 4·0198 ÷ ·62.
7. (802·7 × 4·6) ÷ 1·3 of 7; 207·61 + 1·98 ÷ 7·15.
8. $\frac{12{\cdot}346 \times {\cdot}017}{2{\cdot}7 + 19{\cdot}8}$; $\frac{62{\cdot}704 + 3{\cdot}001 - {\cdot}987}{4{\cdot}1235 - {\cdot}68}$.
9. $\frac{21{\cdot}307 \times 6{\cdot}819}{2{\cdot}3 \text{ of } 7{\cdot}98}$; $\frac{(40{\cdot}6 + 7{\cdot}1) \times (3{\cdot}029 - {\cdot}1874)}{6{\cdot}27 + 8{\cdot}53 - 7{\cdot}1}$.
10. Divide 3·14159 separately by 158, by 70·05, and by ·613.
11. Divide the product of 18·63 and 2·057 by the sum of 19·2 and 231·09.
12. Find the difference between the fifteenth and the sixteenth parts of 297·6832.
13. Express decimally the quotient, if the sum of the numbers $7\frac{2}{3}$ and $8\frac{1}{7}$ be the divisor, and that of $6\frac{2}{5}$ and $15\frac{1}{4}$ the dividend.
14. Divide $230\frac{5}{19}$ by $15\frac{2}{7}$, giving the answer in decimals.

EXERCISE LXXXVIII.

☞ Work the last six sums in exercise LXV. so as to obtain the answers in a decimal form.

1. Add together the sum, the difference, the product, and the two quotients of ·04 and ·005.

2. What fraction should be multiplied by $3\frac{1}{6}$ of $3\frac{1}{8}$ to produce 3?

3. If $1\frac{3}{4}$lb. of sugar cost ·03515625 of £2 16s., what is the value of ·3125 cwt.

4. Find the value of $3\frac{2}{5} + 4\frac{1}{8} + 1\frac{1}{40} + 3\frac{13}{625}$, by vulgar fractions and by decimals, and show that the two results coincide.

5. If the price of an ounce of coffee be $\cdot458\dot{3}$ shillings, what is the value of ·0015625 of a ton?

6. What part of £2 is four-fifths of half a guinea?

** CONTRACTED METHODS OF MULTIPLICATION AND DIVISION.

288. In many problems in Multiplication or Division of Decimals the answer is only required to be accurate as far as a certain decimal place. In such cases some time would be wasted if we were to work out the whole sum according to the rules just given. For example, suppose it is required to find the product of 23·7286 and 414·793: it appears by (279) that the answer will extend to the seventh place of decimals, one of the factors having 3 and the other 4 places. But if for the purpose in hand it is only necessary to have an answer true to the third place, it is evident that some part of the operation is useless, and might be avoided.

I.		II.	III.
414·793		414·793	414·793
23.7286		23·7286	6827·32
248	8758	249	829586
3318	344	3318	1244379
8295	86	8296	290355
290355	1	290355	8296
1244379		1244379	3318
829586		829586	249
9842·457	1798	9842·457	9842·457

In Example I. the figures to the right of the vertical line are unnecessary, as the answer is not required to extend beyond the third place.

In Example II. no part of the multiplication has been performed which is not needed in the answer. Thus in the first line, to have multiplied the final figure 6 by that above it (3) would have produced an answer in the seventh place. But as the answer required is to be in the third place, we multiply the 6 by that figure of the multiplicand which will give a product in the third place. This figure is four places to the left of the 3, and we therefore begin the sum by taking 6 × 1. If we only did this, however, the answer would not be correct, for on looking at the two next figures of the multiplicand we see that the number 28 would come in the next place. But (271) because 30 is nearer 28 than 20 is, we carry 3 into the thousandths place. In the line below it we observe that the number of ten thousandths was 63, we therefore only carry 6 into the thousandths place and neglect the 3. In the third line the number of ten thousandths is 18. We call this 20 rather than 10, and so carry 2 to the thousandths place. Proceeding in this way we avoid the trouble of setting down figures which do not contribute to the answer.

In Example III. the same process is somewhat more conveniently effected by the inversion of the figures of the multiplier.

CONTRACTED RULE FOR DECIMAL MULTIPLICATION.

289. Begin with the right-hand figure of the multiplier, and multiply by it that digit of the multiplicand which will give a product in the place required. Find what would have been carried from the place to the right of it, and add this to the first product.

Take the second figure of the multiplier, and find the product of it and the digit of the multiplicand which stands to the right of that which was formerly chosen. Put the unit of this product into the required decimal place after accounting for the remainder from the right as before.

At each line take one more figure of the multiplicand into the product; proceed as before, and add up the results as in ordinary Multiplication.

EXERCISE LXXXIX.

☞ (a) Find the following products to the third decimal place:—

1. 17·302 × ·579; 60·852 × 19·7; ·203 × 17·98.
2. 106·728 × 3·1957; 72·49 × 10·87632; 4·1985 × 2·1743.
3. 70·96 × 8·074; 32·75 × 41·7209; 62·8145 × 3·172.
4. 7·0968 × 3·12; 7·145 × 6·1437; 8·179 × ·2146.
5. 7·986 × 2·09 × 4·723; 6·798 × 4·072 × ·5.
6. 5·189 × 64·3274; 8·274 × 35·2968.
7. 4·1725 × 31·72; 4·096 × ·38 × 1·792.

(b) Find the products of the first six sums in Exercise LXXXIV., contracting each answer by two digits.

290. DIVISION.—When the quotient of a Division sum is only required to be true to a certain decimal place, a similar method may be adopted.

Example.—Divide 7·9362 by 2·7451 to four places of decimals.

Uncontracted method.

```
I.  2·7451)7·9362(2·8910
           54902
           -----
           24460|0
           21960|8
           -----|---
            2499|20
            2470|59
           -----|---
              28|610
              27|451
              --|----
               1|1590
```

Contracted method.

```
II.  2·7451)7·9362(2·8910
            54902
            -----
     2·745)24460
           21960
           -----
      2·74)2499
           2470
           ----
        2·7)28
            27
            --
             1
```

The figures to the right of the vertical line, in Example I., are not necessary in order to obtain the required result. As the answer is only to be true to the fourth place of decimals, the reduction of the remainders into equivalent values in the fifth, sixth, and seventh places is unnecessary. In Example II., instead of adding a cipher to the first remainder, to reduce it to a lower denomination, we deal with it by itself, *cutting off the last figure of the divisor*, and applying the new quotient (8) only to the first four figures of the divisor. In like manner, when the next remainder is found we do not reduce it, but apply the next partial quotient only to the first three figures of the divisor.

Here, as in the former instance, we mentally determine what figure would have been brought from the place to the right if the sum had been uncontracted; this number is added in the multiplication.

CONTRACTED RULE FOR DECIMAL DIVISION.

291. Begin as in ordinary Division, and having obtained the first figure of the quotient, cut off the last figure of the divisor, instead of bringing down the next figure of the dividend or adding a cipher. At each step cut off one figure of the divisor. Add in at each multiplication the number which would have been carried from the next place to the right.

In the example just given, five figures were required in the answer (*i. e.*, the integer and four decimal places), and therefore the whole five figures of the divisor were treated in order to obtain the first figures of the quotient. But if the divisor had contained six or more digits, all except the first five might have been neglected altogether. If only two decimal places had been required in the answer, it would have been sufficient to deal with the first three figures of the divisor. In all such sums, therefore, we take as many figures of the divisor as will be needed in the quotient, and neglect the rest.

EXERCISE XC.

☞ (a) Find the following quotients to the third decimal place :—

1. 157·038 ÷ 79·618 ; 26·814 ÷ 7·29 ; 34·076 ÷ 18·729.
2. 7·41036 ÷ 219·6 ; 32·079 ÷ 68454·7 ; 13·72 ÷ 4·196.
3. 14·723 ÷ 6·18 ; 2·098 ÷ 63·572 ; 14·073 ÷ 2·198.
4. ·0071 ÷ ·6937 ; 8·12473 ÷ 6·2389 ; 51·247 ÷ ·813765.
5. 3·5689 ÷ ·27432 ; ·016541 ÷ ·7198 ; 31·275 ÷ ·68554.

(b) Find the quotient of the sums in Exercise LXXXVII., so that the answers shall be true to the fourth place.

SECTION X.—APPLICATION OF DECIMALS TO CONCRETE QUANTITIES.

292. CASE I.—TO REDUCE A QUANTITY TO A DECIMAL OF A LOWER DENOMINATION.

RULE.

Multiply by as many of the less as make one of the greater, pointing off as many decimal places in the answer as there are in the multiplicand.

Example.—What decimal of a penny is ·69 of £1? By (232) the answer to this question must be as much greater than ·69 as £1 is greater than a penny. Hence £·69 = 20 × ·69 of a shilling = 20 × 12 × ·69 of a penny.

$$\begin{array}{rl} \cdot 69 & \text{of } £1 \\ 20 & \\ \hline 13\cdot 8 & \text{of 1s.} \\ 12 & \\ \hline 165\cdot 6 & \text{of 1d.} \end{array}$$

293. CASE II.—TO REDUCE A QUANTITY TO A DECIMAL OF A HIGHER DENOMINATION.

RULE.

Divide the decimal, step by step, by as many of the less as make one of the greater, until the required denomination is reached.

Example I.—What fraction of 1 cwt. is ·798 of a pound? Now (232) ·798 of 1 lb. = ·798 of $\frac{1}{112}$ of 1 cwt. = $\frac{798}{1000} \times \frac{1}{112}$. But ·798 ÷ 112 = ·007125. Answer, ·798 of 1 lb. = ·007125 of 1 cwt.

Example II.—What fraction of a week is 18·56 of an hour?

$$\begin{array}{rl} 24\underline{)18\cdot 56} & \text{of an hour.} \\ 7\underline{)\cdot 7733} & \text{of a day.} \\ \cdot 11047 & \text{of a week.} \end{array}$$

Answer, 18·56 hours = ·7733 of a day = ·1104714 of a week.

EXERCISE XCI.

1. Express ·37 of a pole in terms of a league, a mile, a furlong, a yard, a foot, and an inch.

2. Express ·75 of a shilling as decimals of each of the coins of the realm.

3. What fraction of an hour is ·58 of a week? and how much of a day is ·074 of a minute?

4. Express ·784 of a bushel as fractions of a quarter, and of a peck.

5. Express ·78 of 1s. 3d. as fractions of 3d., of 2s. 6d., and of 15s.
6. How much is 14·973 shillings of a ten pound note? and how much of a farthing?
7. Reduce 3·35 shillings to the decimal of £1, of 10s., and of ½d.
8. Express ·5 shilling + ·7 crown + £·15 as a decimal of a florin.
9. What fraction of 11 cwt. is ·297 lbs.?
10. How much of a bushel is ·397 of a pint?
11. Reduce 1·27 oz. to decimals of a grain, a scruple, and a dwt.
12. How much of a square mile is ·39 of a rood?

294. CASE III.—TO FIND THE EQUIVALENT VALUE TO A GIVEN DECIMAL IN ORDINARY CONCRETE NUMBERS.

RULE.

Reduce only the fractional part of the expression as in Case I., reserving the whole number as part of the answer. Continue to reduce the fractions until the lowest denomination is reached. The series of whole numbers will form the answer.

Example I.—What is the actual value of 23·568 of 1s.? Here 23·568 of a shilling = 23 shillings + ·568 of 1 shilling.

But by (292) ·568 of 1s. = ·568 × 12 = 6·816 of 1 penny.
= 6 pence + ·816 of a penny.

Again, ·816 of a penny = ·816 × 4 = 3·264 of a farthing.
= 3 farthings + ·264.

Hence 23·568s. = 23s. + 6·816d. = 23s. 6d. 3·264 farthings.
= £1 3s. 6¾d. + ·264 farthing.

Example II.—Reduce 29·3814 cwt. to the ordinary form.

$$\begin{array}{r l} 29\cdot3814 & \text{cwt.} \\ \underline{4} & \\ 1\cdot5256 & \text{qrs.} \\ \underline{28} & \\ 42048 & \\ \underline{10512} & \\ 14\cdot7168 & \text{lbs.} \\ \underline{16} & \\ 11.4688 & \text{ounces} \end{array}$$

Answer, 29 cwt. 1 qr. 14 lbs. 11·468 oz.

Example III.—Reduce ·178 of a mile to the ordinary form.

$$\begin{array}{r l} \cdot178 & \text{of a mile} \\ \underline{8} & \\ 1\cdot424 & \text{furlongs} \\ \underline{40} & \\ 16\cdot960 & \text{poles} \\ \underline{5\frac{1}{2}} & \\ 5\cdot280 & \text{yards} \\ \underline{3} & \\ \cdot840 & \text{feet} \end{array}$$

Answer, 1 furlong, 16 poles, 5·84 feet

EXERCISE XCII.

☞ Reduce the following to the ordinary forms of expression :—

1. 17·914 weeks; ·178 gallons; 32·746 weeks.
2. 2·074 guineas; £5·728; 16·474 shillings.
3. £17·5028; £1·627; £4·78.
4. £·68; £2·47; £1·28.
5. £103·4792; £2·18634; £7·096.
6. £7·8425; £303·796; £2·0872.
7. 29·324 cwt.; 5·87 cwt.; 2·97 quarters.
8. ·07 of £2 10s.; £·972916; £·10875.
9. 3·27 days 4·728 weeks; 39·278 hours.
10. 1·29 miles; 27·28 furlongs; 3·05 leagues.
11. 27.38 yards; 4·196 poles; 282·72 miles.
12. 37·25 acres; 48·972 acres; 25·34 sq. roods.
13. 13·38 sq. yards; 14·23571 sq. miles; ·978 acres.
14. 17·84 gallons; 212·305 gallons; 4·07 quarts.
15. 29·732 quarters of corn; ·08 quarters; 53·197 quarters.
16 18.7 English ells. 3·207 French ells. 132·72 nails.

295. CASE IV —WHEN A CONCRETE NUMBER IS GIVEN IN THE ORDINARY FORM, TO EXPRESS IT DECIMALLY—

RULE.

Begin with the quantity of the lowest denomination named. Treat this separately, and divide it by as many of the less as make one of the next higher name. Add the whole number of this higher name to the fraction thus found, and divide again so as to reduce this to the denomination next above. Continue to divide and to add in the whole numbers one by one, until the required denomination is reached.

Example I.—Express 2 lbs. 7 oz. 3 dwt. 15 grains as decimals of 1 lb. troy.

Here (233) 15 grains = $\frac{15}{24}$ = ·625 of 1 dwt.
∴ 3 dwt. 15 grains = 3·625 dwt.
But 3·625 dwt. = $\frac{3{\cdot}625}{20}$ = ·18125 of 1 oz.
∴ 7 oz. 3 dwt. 105 grains = 7·18125 oz.
Again 7·18125. = $\frac{7{\cdot}18125}{12}$ = ·5984375 of 1 lb.
∴ 2 lbs. 7 oz. 3 dwt. 15 grains = 2·5984375 lbs.

The same sum would be more conveniently worked in this form :—

2 lbs. 7 oz. 3 dwt. 15 grains
24)15 grains.
20)3·625 dwt.
12)7·18125 oz.
2·5984375 lbs.

Example II.—Reduce £17 9s. 8¾d. to the decimal of 1s.

4)3 farthings
12)8·75 pence = 8¾d.
9·72916 shillings = 9s. 8¾d.
340· shillings = £17
349·72916 shillings = £17 9s. 8¾d.

Example III.—Reduce 7 qrs. 5 bush. 3 pecks 1 gal. 2 quarts to the decimal of a quarter.

4)2 quarts
2)1·5 gallons = 1 gal. 2 qts.
4)3·75 pecks = 3 pecks 1 gal. 2 qts.
8)5·9375 bushels = 5 bush. 3 pecks 1 gal. 2 qts.
7·7421875 quarters = 7 qrs. 5 bush. 3 pecks 1 gal. 2 qts.

EXERCISE XCIII.

1. Reduce £17 10s. 9¾d., £2 5s. 7½d., and £8 15s. 4¼d. to the decimal of £1.

2. Reduce £25 18s. 7½d., £1 10s. 5¾d., and £297 10s. 7½d. to the decimal of £1.

3. Reduce 17 cwt. 3 qrs. 2 lbs., 3 qrs. 9 lbs., and 427½ lbs. to the decimal of 1 cwt.

4. Reduce 294 cwt. 3 qrs. 17 lbs., 10 cwt. 27 lbs., and 17 cwt. 9 lbs. to the decimal of 1 qr.

5. Reduce 17 lbs. 6 oz., 2 qrs. 1 lb., and 3 cwt. 1 qr. to the decimal of 1 lb.

6. Reduce 13 oz. 4 drs., $29\frac{3}{7}$ lbs., and 4 cwt. 2 qrs. 7 lbs. 4 oz. to the decimal of 1 lb.

7. Reduce 5 days 12 hours 25 mins. 37 secs. to the decimal of a week.

8. Reduce 3 weeks 3 days 23 hours to the decimal of a week.

9. Reduce 12 hours 55 mins. 23⅓ secs., and 17 hours 47 mins. 33 secs. to the decimal of a day.

10. Reduce 7 oz. 4 dwt. to the decimal of a pound troy, and 9 oz. $2\frac{3}{7}$ drs. to the decimal of a pound avoirdupois.

11. Reduce 8s. 7¼d. to the decimal of a guinea, and £3 17s. 6d. to that of a shilling.

12. Reduce 19 qrs. 2 bush. 2 pecks to the decimal of a quarter, and 7 quarts 3 pints to that of a gallon.

SECTION XI.—DECIMAL MONEY.

296. The Arithmetic of concrete quantities is more difficult than that of abstract numbers, because the units are not subdivided decimally, but follow in each case some special rule. Thus the principal units of weight, of time, of linear measure, of value, are variously subdivided in the following manner:—

1 cwt. = 4 qrs. = 112 lbs. = 1792 ounces.
1 week = 7 days = 168 hours = 10080 minutes.
1 mile = 8 furlongs = 320 poles = 1760 yards
= 5280 feet = 63360 inches.
£1 = 20 shillings = 240 pence = 960 farthings.

In the chapter on Money, Weights, and Measures (Appendix A) some reasons will be found for these irregular divisions; but it is manifes that if we had been accustomed to divide each of these principal unit into tenths, hundredths, thousandths, &c., the addition, subtraction, multiplication, and division of concrete numbers would be much easier than it now is. If, for example, we had a name for the tenth and hundredth of a ton, instead of the twentieth and the eightieth, we should be spared the trouble of asking ourselves a question in reduction at every step of a sum in avoirdupois weight, and the whole process would exactly resemble simple Arithmetic. But long-established usage and habits have made it very difficult to alter the familiar weights and measures which are current among us, and such a complete decimal system as has been adopted in France, and is described in the Appendix, though it has been much discussed in this country, is not likely to come into general use, if at all, for many years.

297. It has been pointed out that our coinage could be decimally adjusted with less difficulty than any other concrete quantity. The tenth of a pound is 2 shillings; and although we have no exact equivalent in our present coinage for the hundredth of a pound, yet the farthing, which is $\frac{1}{960}$ of a pound, differs very little from $\frac{1}{1000}$. If therefore we retained the pound as the unit, three things would become necessary—I. To give a name to the sum now called 2 shillings, and to make this the second column in our accounts. II. To have a new coin and a new name for the tenth of 2 shillings, or the hundredth of a pound, and to make this the third column of our accounts; and III. To depreciate the value of the farthing 4 per cent., so that it shall become $\frac{1}{1000}$ of the pound, and rank in the fourth place of our accounts.

On such a system the present irregular division of our money by 20, 12, and 4 would disappear, and would be replaced by the following:—

1 MIL = one-thousandth of £1.
1 CENT = 10 mils = one-hundredth of £1.
1 FLORIN = 10 cents = 100 mils = one-tenth of £1.
1 POUND = 10 Florins = 100 Cents = 1000 Mils.

298. Such an expression as £17·387 would then mean £17, 3 florins, 8 cents, 7 mils, and might be multiplied and divided as readily as the number 17,387; because ten of any one place would equal one in the place to the left.

299. But although such a coinage may not be adopted, some of the advantages of the decimal system for purposes of computation may easily be obtained. This will be seen from the following considerations. Taking £1 as the unit—

s.	*d.*		*s.*	*d.*	
10	0	= ·5	1	0	= ·05
5	0	= ·25	0	6	= ·025
2	6	= ·125	0	1	= ·0041$\dot{6}$
2	0	= ·1	0	0¼	= ·001041$\dot{6}$

Of these several values some are much more easily expressed as decimals of £1 than others. All except the penny and the farthing might be retained in a decimal system. The relation between these two coins and £1 cannot be expressed by the help of 10 or any power of 10. But as the farthing is now $\frac{1}{960}$ of £1, we may represent it as ·001 of £1 without serious error, except in cases where the decimal expression has to be multiplied considerably.

300. When accuracy is only required to the third place of decimals it is sufficient to remember the following short table:—

s.	*d.*		*s.*	*d.*	
2	0	= ·1	0	6	= ·025
1	0	= ·05	0	0¼	= ·001

Hence £21 18s. 7¾d. may be written as 21 + ·9 + ·025 + ·007 = 21.932. We take 9 in the first place to represent 18s., ·25 in the second and third places for the 6d., and 7 for the additional farthings.

Suppose, now, it were required to *divide* this sum by 6, the work would be done very readily thus :—

$$\begin{array}{l} 6\underline{)21{\cdot}932} \\ \quad 3{\cdot}655 \end{array}$$ = £3, 6 tenths, 5 hundths., 5 thoudths., = £3 13s. 1¼d.—

the decimal expression being transformed into £3, 12s., 1s., 5 farthings, or £3 13s. 1¼d. But if it had been required to *multiply* this sum of money by 6, it is evident that whatever error is made in calling 7 farthings ·007 of £1 is also multiplied by 6. Hence the following rules may be safely adopted, only :—

(1) Whenever the sum of money to be dealt with contains no lower fraction of £1 than 6d. ; *e.g.*, generally in Interest sums.

(2) Even though the fraction be less than 6d. ; whenever the problem is one in division, subtraction, or short addition.

But the rule cannot be safely employed when a sum of money is expressed in lower fractions of £1 than 6d., and is required to be multiplied, for then, the error is increased.

301.—CASE I.—TO REDUCE ANY SUM OF MONEY TO A DECIMAL EXPRESSION WHICH SHALL BE TRUE TO THE THIRD PLACE—

RULE.

Take the pounds as whole numbers and fill up the three places of decimals as follows :—100 for every 2 shillings, 50 for a shilling, 25 for 6 pence, and 1 for every additional farthing.

Example I.—£153 15s. 4¼d. = 153·767. Because 14s. = ·700, 1s. = ·050, and 17 farthings = ·017, and 700 + 50 + 17 = 767.

EXERCISE XCIV.

☞ Reduce the following sums to decimals true to the third place :—

1. £2 14s. 8d. ; £3 10s. 9½d. ; £27 4s. 6¾d.
2. £7 18s. 4½d. ; £235 15s. 3d. ; £228 6s. 7½d.
3. £14 17s. 9d. ; £628 4s. 7d. ; £5 13s. 4¼d.
4. £1 11s. 7½d. ; £274 16s. 3d. ; £2 0s. 9¾d.
5. £2 7s. 4¼d. ; £8 11s. 2¾d. ; £17 6s. 9½d.
6. £8 2s. 4½d. ; £37 2s. 3¼d. ; £11 9s. 4¼d.

CASE II.—TO CHANGE A DECIMAL EXPRESSION INTO THE ORDINARY FORM—

RULE.

Take two shillings for every 1 in the first decimal place, one shilling for every 5 in the second place, and a farthing for each of the remaining figures to the third place; unless the number be 25 or over, in which case omit one farthing from the answer.

Example.—£19·672 = £19 13s. 5½d. For

·6 = 12s. 0d.
·05 = 1 0
·022 = 0 5½
·672 = 13 5½

EXERCISE XCV.

☞ Reduce the following decimals to ordinary numbers:—

1. £19 8 tenths 6 hundredths 4 thousandths; £25 3 tenths 7 hundredths; £176 1 tenth 3 hundredths 5 thousandths.
2. £17·528; £1·064; £18·371; £20·965; £8·27.
3. £16·721; £4·128; £3·074; £4·106; £2·093.
4. £4·696; £18·325; £71·421; £58·372; £41.629.
5. £7·284; £5·901; £8·623; £5834·721; £209·645.
6. £·872; £3·96; £·047; £814·756; £203·871.
7. £·97; £·083; £·46; £8·695; £23·72; £·072.

EXERCISE XCVI.

☞ Work the first four sums in Exercise XII.; the first six in Exercise XXIX.; and the first three in Exercise XXX., by this method.

302. CASE III.—WHEN IT IS REQUIRED TO MAKE THE EXPRESSION ACCURATE TO ANY GIVEN PLACE OF DECIMALS—

RULE.

Express the sum of money as far as sixpence by the rule just given; but for every penny above an even sixpence place $41\dot{6}$ in the third and following places, and for every odd farthing place $1041\dot{6}$ in the third and following places, as far as required.

303. *Observation.*—The table given in (299) shows that the former rule gives perfectly accurate results as far as 6d., but that the value of odd pence and farthings cannot be expressed without descending lower in the decimal scale. As in both cases repeating decimals occur it is impossible to be entirely accurate even then, but it is easy to carry the calculation so low that the error shall be as slight as we please.

Example.—Reduce £17 13s. 8d. to decimals true to the fifth place, and £270 3s. 2¾d. true to the seventh place.

	£	s.	d.		£.
I.	17	0	0	=	17
	0	12	0	=	·6
	0	1	0	=	·05
	0	0	6	=	·025
	0	0	2	=	·0083$\dot{3}$
					17·68333

	£	s.	d.		£.
II.	270	0	0	=	270
		2	0	=	·1
		1	0	=	·05
	11 × ·0010416				·0114583
					270·1614583*

EXERCISE XCVII.

☞ Reduce the following to decimals true to the sixth place:—

1.	£17 10s. 3½d.;	£27 13s. 4¾d.;	£109 8s. 6½d.
2.	£273 5s. 1¼d.;	£12 6s. 8¾d.;	£1 6s. 3¼d.
3.	£28 9s. 4¼d.;	£1 0s. 7½d.;	£23 5s. 6¾d.
4.	£17 0s. 8½d.;	£29 16s. 3¼d.;	£18 0s. 9d.

EXERCISE XCVIII.

☞ Work the first four sums in Exercise VIII., and in Exercise XX., by this method.

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* This process, though apparently long, is much simpler when performed mentally than when laid out at length on paper. A very little practice will enable a student to use both rules with great facility.

Questions on Decimal Fractions.

What is meant by the expression Decimal Fraction? In what respect does this Fractional Notation resemble the common Notation of Integers? What is the main difference between the two? What purpose is served by the number ten in Integral Notation? what in Fractional Notation?

In the following line of figures what is the separate value of each, 7928 ·534728? Compare the values of two figures, one of which is 5 places to the right and the other 5 places to the left of the decimal point. Describe the use of the cipher in Fractional Notation. Why are decimal fractions very readily multiplied or divided by ten or by any power of ten? Give the rule which applies to such cases.

What is the great advantage possessed by decimal over vulgar fractions? Has a decimal fraction any denominator? if so, how do you determine what it is? What process in Decimals is analogous to the reduction to a common denominator of Vulgar Fractions? Describe it, and state the rule. How are decimals converted to vulgar fractions, and vulgar to decimals? Give the reason for the rule in each case. Show in what way this rule resembles that of ordinary Division or Reduction.

What are Recurring Decimals? How many kinds of them are there? In what cases do they occur? How may they be reduced to vulgar fractions? Demonstrate the reason of the rule. What kind of vulgar fractions always give recurring decimals?

How do you determine the position of the point in the answer to a sum in Addition or Subtraction of Decimals? Why? In what way do the Addition and Subtraction of Decimals differ from, and in what way do they resemble, the Addition and Subtraction of Integers?

Where should the decimal point be placed in a product or in a quotient? Why? Show how the Rule for Multiplying Vulgar Fractions applies to Decimals. What is there in Decimals corresponding to the multiplication of the denominator in Vulgar Fractions?

When is Division of Decimals the same as in whole numbers? What general rule would apply equally to all possible cases of Division of Decimals? Why is not the same rule applied in all cases? State the different special rules of Division, and give a reason for the use of each. How does the rule for Decimal Division correspond to that in vulgar fractions for inverting the terms of the dividing fraction? Give an example.

Why is it better to contract certain sums in Multiplication and Division? State the processes, and show when each should be used. What sort of error, if any, is liable to occur in these processes?

State the axiom assumed in the rule for reducing a certain value into the decimal of a greater or a less. State the rule in each case, and give an example showing how the axiom applies. If a certain fraction of a Troy pound had to be reduced to the decimal of an Avoirdupois pound, what rule would be necessary?

What would be the advantages of decimalizing our coinage. Describe a possible system founded on £1 as the unit. What is the readiest way of converting our present money into decimals of £1? What sort of error will occur in using the rule, and when?

MISCELLANEOUS EXERCISES ON FRACTIONS.

1. What is the approximate value of £54·732 and £763·824?

2. Add together $3\frac{1}{8}$, $7\frac{1}{3}$, 25·687, and 1·666.

3. A can do $\cdot\dot{6}$ of a piece of work in 4 hours, B can do ·75 of the remainder in one hour, and C can then finish it in $\frac{1}{3}$ of an hour; in what time can A, B, and C do it together?

4. Add together $\frac{1}{3}$ of $\frac{5}{7}$; $\frac{1}{4}$ of $\frac{12}{21}$; and $\frac{5}{6} \div 1\frac{5}{9}$.

5. Reduce 3s. $4\frac{1}{2}$d. to the fraction and to the decimal of £1.

6. Reduce ·4786 days to hours, minutes, and seconds; and find the value of 173·25 yards at 5·25s. per yard.

7. Reduce $\frac{2}{3}$ of a day to the decimal of a week and of a year.

8. Simplify each of the following expressions:—

$$\frac{9\frac{5}{7}}{13\cdot25};\quad \frac{\frac{2}{3}\text{ of }\frac{4}{5}}{\cdot15+\frac{3}{8}};\quad \frac{4\frac{4}{7}}{2\cdot3};\quad \frac{\cdot284}{\frac{2}{7}\times\frac{4}{5}};\quad \frac{6\frac{5}{7}}{2\cdot75}.$$

9. What is the difference between $\frac{2}{3}$ of $\frac{4}{7}$ of a pound, and $\frac{5}{7}$ of a shilling?

10. Add together $\frac{1}{7}$ of a pound, $\frac{2}{3}$ of a shilling, and $\frac{5}{12}$ of a penny.

11. Add together $\frac{2}{3}$ of a pound, $\frac{3}{14}$ of a guinea, and $\frac{2}{3}$ of 4s. 7d.

12. Multiply and divide 60 by ·00048, also 7·29 by ·0028.

13. If to one person a testator bequeaths $\frac{9}{20}$ of his property, to another $\frac{1}{4}$, and to another £300, what is the value of that property?

14. Add together the circulating decimals 0·53434, &c., and ·0465858, &c., and subtract the sum from $1\frac{1}{3}$.

15. Subtract $\frac{3}{7}$ from 1·1, and divide the remainder by 0·1.

16. What is the value of 72 lbs. 3 oz. at £4 3s. 2d. per cwt., and of 6 cwt. 1 qr. 19 lbs. at £7 10s. per cwt.?

17. Find a series converging to $\frac{251}{764}$, and also to 3·14159.

18. What is the rent of 311 acres 2 roods 26 poles at £1 2s. 9d per acre?

19. If a vessel is two-thirds full, and after 70 pints are drawn off is found to be three-eighths full, how much could it contain?

20. If ·7 of anything cost £29 8s. 7d., what share will be worth £15 10s. 4d.?

21. Find the greatest common measure of 236·511 and 37·499, also of 20·75 and 11·39.

22. The circumference of a circle is 3·14159 of the diameter; find the circumference of circles whose diameters are 13·7 feet, 1·96 yards, and 28·342 miles respectively.

23. What are the diameters of circles whose respective circumferences are 12·56 inches, 3·297 feet, and 11·08 miles?

24. Find a quarter's rent on 114·76 acres at £3·74 per annum per acre, and two years' rent on 47·5 acres at the same rate.

25. Work the following sums decimally, true to three places—

(a). 1172 at £1 8s. 9d.; 3274 at £7 6s. 6d.; 2093 at 16s. 7½d.

(b). 2093·5 at £7 12s. 4d.; 812·53 at 17s. 6d.; 412·9 at £1 2s. 1d.

(c). 81·7 at £4 10s. 3¼d.; £519·6 at £17·28; 327·4 at £·68.

(d). 41·5 at £2·78; 8062 at £12·093; 712·34 at £·06.

26. If a man spends $\frac{5}{7}$ of his income in board and lodging, and ·125 in dress and amusements, and then can save £117 per annum, what is his income?

27. If a bath be supplied with water by two pipes, one of which alone would fill it in 20 and the other in 15 minutes, and if a discharging pipe would empty it in 25 minutes, how long will it be in filling if all three pipes are employed together?

28. If seven persons take equal portions of a piece of cloth 19 yards long; express the share of each in decimals of an English ell.

29. If, in an election, 1,253 voted, and one candidate had a majority of 89 over the other, what fraction of the whole voted for each?

30. A French metre is 39·371 English inches. Express the following lengths in metres: 3 miles 7 furlongs 12 poles, 5 miles 1250 yards, 17 miles 3 furlongs 2 yards 15 inches, and 7·2854 furlongs.

31. Multiply the square of $1\frac{5}{76}$ by $3\frac{15}{47}$, and give the answer in decimals.

32. If two persons do work in ·10·5 days which one of them alone would finish in 17·75 days, in what time would the other do it?

33. A gallon contains 277·274 cubic inches. Find the cubic contents of the following measures,—a gill, a pint and a half, a measure containing 3 quarts, and an 11 gallon cask.

34. When the rate of exchange is such that an English sovereign is worth 25·19 French francs. how would the following sums be expressed (most nearly) in French money?—£3 17s. 10½d., £18 6s. 8d., and £214 7s. 4¼d.

35. At the same rate what would be the nearest English equivalents to 17·96 francs, 2183·45 francs, and 1069·4 francs?

36. If an ounce of gold be worth £$4{\cdot}1895\dot{3}$, what is the value of $375\ \dot{3}$ lbs.?

37. Compare the number of revolutions made by two wheels, one of which is 3 feet and the other 5 feet in diameter, in rolling over 17 miles.

38. Three-sevenths of an estate is sown with wheat; ·15 of it is meadow land, and the rest consists of 24 acres 2 roods 17 poles; find the whole area of the land.

39. If A owns ·48 of a ship and B the rest, and the difference in the value of their shares is £23·76, what is the worth of the ship?

40. If $\frac{5}{7}$ of an estate contains 250 acres, and is worth £1003 17s. 1d., what is the extent and value of ·15 of it?

41. A Spanish dollar is worth 3·3 shillings. Express 515¾ Spanish dollars in English money, and £728 13s. 4d. in Spanish.

42. The specific gravities of gold, silver, and copper, are respectively 19·36, 10·47, and 8·95. Find the weights of a lump of each, of the same bulk as a vessel of water weighing 7 lbs. 6 oz.

43. The year consists of 365·24224 days. In what time will the error of calling it 365¼ days amount to an error of one day?

44. The area of a circle is found by multiplying the square of the diameter by one-fourth of 3·14159. Find the areas of circles whose diameters are respectively 12·16 feet, 73 inches, and 2·75 yards.

45. In the event of a change of coinage, the rate of postage would be necessarily altered from one penny either to 5 mils or to 4 mils. Compare the number of letters which could be sent for £47 11s. 6d. at the three several rates of 5 mils, one penny, and 4 mils.

46. £25 3s. 6d. are now received on an average per week, at a turnpike where 1½d. is charged for every vehicle. What would the toll-contractor lose per week if the toll were lowered to 6 mils, and what would be the difference in his receipts if it were altered to 7 mils?

47. In a public school 70 boys on an average pay 2d. per week throughout the year, and 55 boys pay 3d.; what are the annual receipts? and what would they be if the former paid one cent. and the latter one cent and a half instead?

48. The length of a seconds pendulum is 39·1393 inches; express this as a fraction of a foot, of a yard, and of a furlong.

49. The circumference of a circle measures 3·14159 times its diameter: what length of railing will be sufficient to surround three circular plots of ground whose diameters are 27·8, 19·16, and 32·081 feet respectively?

50. The imperial gallon contains 277·274 cubic inches: what are the solid contents of a quart, pint, and gill measure respectively?

51. What is the difference between the average summer and winter temperature of Quebec, the former being 68·08°, and the latter 14·15° Fahrenheit?

52. If a cubic foot of water weigns 1,000 ounces, what is the weight of water in a vessel which holds 7 gallons $3\frac{1}{2}$ pints?

53. A ship, being captured by 6 officers and 30 sailors, had on board specie to the amount of £3,000; the officers claimed $\frac{2}{5}$ of the sum; what was the share of each sailor?

54. Find the sum, difference, and product of $\frac{1}{99}$, and $\frac{1}{999}$, and express the answers decimally to seven places.

55. How much will remain of $\frac{7}{9}$ of £25·1, after the following articles have been paid for, viz., $1\frac{4}{5}$ yards of cloth at £$\frac{4}{5}$ per yard, $17\frac{7}{8}$ yards of calico at $\frac{3}{8}$ of 1s. per yard, and $12\frac{1}{5}$ yards of linen at $\frac{1}{8}$ of £1 per yard?

56. Three pipes empty a vessel separately in 6, 4, and 2 hours; how long will it take to empty the same vessel if all are used together?

57. Express $\frac{5}{6}$ of $\frac{3}{8}$ of 13s. 4d. as a decimal of a crown.

58. Find the value of 37,461 articles at 3s. $10\frac{3}{4}$d. each, and also of the same number of articles at 8s. $10\frac{3}{4}$d.

59. If A can finish a piece of work alone in 4 days, and B in 6 days, in what time will both do it together?

60. 7,000 grains make 16 avoirdupois ounces, and 5,760 grains make 12 troy ounces: what is the difference in grains between 9 ounces troy and 11 ounces avoirdupois?

61. Reduce $3\frac{3}{4}$ crowns to the fraction of £1 12s. $9\frac{3}{4}$d., and $2\frac{2}{3}$ half-guineas to the fraction of 10s. $11\frac{1}{4}$d.

62. Express the following as decimals of a pound :—

$\frac{3}{17}$ of £20; $\frac{5}{9}$ of £25 10s.; $\frac{12}{27}$ of £47 3s. 6d.

63. What length added to $\frac{7}{9}$ of a mile would amount to ·865 of a league?

64. From what weight can I subtract ·27 of a cwt. so as to leave ·06 of a ton?

65. What number multiplied by the product of $\frac{3}{7}$ and $\frac{5}{8}$ will equal the difference between ·05 and ·18?

66. What number divided by the sum of 16·3 and $4\frac{1}{7}$ will give $102\frac{5}{8}$ as quotient?

67. If $28\frac{4}{17}$ be the divisor, and 9·263 the quotient, what is the dividend?

RATIO AND PROPORTION.

SECTION I.—THEORY OF THE PROPORTION OF ABSTRACT NUMBERS.[*]

304. From (3) it appears that we possess no ideas of *absolute magnitude* of any kind.[†] All the notions we have of size, of weight, or of duration, or of any magnitude whatever, are *relative.* To form a clear notion of any such magnitude we must compare it with some one thing of the same kind which we have either arbitrarily chosen or which exists in nature. When we have made this comparison the result is what we call a *number.*

305. All Arithmetic is employed in establishing comparisons between some quantity supposed to be previously known, and some other quantity which is intended to be expressed. Thus, Integral Arithmetic takes unity as the standard of comparison, and considers all other magnitudes as multiples of it. Fractional Arithmetic takes unity as the standard of comparison, and considers it as separated or divided into parts.

But if instead of merely considering the relationship between a certain magnitude *and unity,* we compare any two numbers *with one another,* then the result of such comparison is called the *ratio*[‡] of the two numbers.

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[*] The student who has not read Section IX., on Multiplication and Division, should do so very carefully before commencing this chapter.

[†] The fact that all our notions of magnitude are relative and not absolute will be more clearly seen from the use of such current words, as great, small, many, few, tall, short. Not one of these words has any definite and invariable meaning. By a great house we mean a house greater than average houses. A small castle is one which is small compared with other castles. Yet the former, though properly called great, may be absolutely less in size than the latter. The same number of people which might be called many in a garden would be few in a city. In all our thoughts of magnitude we silently refer to some one familiar standard, and measure by comparison with it.

[‡] I.—The word which Euclid employs here is λόγος (*logos*), which has several

306. There are two ways in which magnitudes of the same kind may be compared with each other:

I. When we ascertain by how much one magnitude exceeds the other. This is to be done by subtracting the less from the greater.

II. When we ascertain how many times one contains the other. This is to be done by dividing the one by the other:

Thus $36 - 4 = 32$, but $36 \div 4 = 9$.

The result of the comparison of these two numbers by the former method is 32; the result of the comparison of the two numbers by the latter method is 9.

307. *Observation.*—Of these two methods of comparison the second is that which naturally suggests itself to the mind, and which we habitually employ. If of two persons one possesses £9,998 and the other £10,000, we should think the difference very trifling, and should look upon them as equally rich; but if one possessed £3 and the other £5, we should not look upon the one sum as near to the other. Yet there is the same absolute difference here as in the former case.

308. Of these two kinds of relationship, the result of comparison by Subtraction is sometimes called the *arithmetical ratio*, and the result of comparison by Division the *geometrical ratio*. These terms, however, are very inappropriate, and will not be again employed here. The word *ratio* when standing alone is generally intended to represent the latter relationship only, and it is in this sense that it will be hereafter used. The comparison of two magnitudes by Subtraction is sometimes called Arithmetical Proportion, and is discussed in the Section on Arithmetical Progression. See Section 500.

309. Ratio* is the relationship which subsists between two quantities of like kinds, with respect to the number of times one contains the other.

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meanings in Greek, but in its application to Mathematics is always used in the limited sense indicated by the definition in the text. The Roman word "ratio" exactly represented its meaning. In English we have no term of the same signification, and therefore we employ this Latin word "ratio" for the purpose. The French use sometimes the word "raison," as equivalent to "ratio," but more frequently the general word "rapport," which is applied to both the kinds of relationship described in the text.

* II. Euclid's definition in Book V., as commonly translated, is apparently more simple than this. "Ratio is the relation between two magnitudes of like kind with respect to quantity." But it is evident that to ask what is the relation of two magnitudes with respect to quantity may be to ask *how much* greater one is than the other; and the answer to this question does not express the ratio of the two numbers

310. Whenever two numbers are compared in this way the first is called the Antecedent, and the second the Consequent, of the ratio.

We express this relation by placing (:) between the two.

Thus 7 : 12 = the ratio of 7 to 12 = the part which 7 is of 12.

Here 7 is the antecedent and 12 the consequent.

311. *Observation I.*—Ratio can only subsist between magnitudes *of like kinds*. The reason which was given (30) why numbers cannot be added or subtracted unless they refer to the same things applies equally here. No two magnitudes can be compared which are not of the same sort. If we were asked to discuss the relation between 7 cwt. and 3 hours, the mind would refuse to entertain the question, and reject it as an absurdity.

312. *Observation II.*—Ratio has just been called the result of the comparison of two quantities or numbers. Before we can make such a comparison we must have a clear notion of the magnitude of each, but when we have made it we obtain a notion of a third magnitude, and this is what is called the "ratio" of the two others. It is quite distinct from the two former magnitudes. If I see one pole 3 yards high, and another 5 yards high, and I compare their heights together, I say that the one is to the other as 3 is to 5, or that the one measures $\frac{3}{5}$ of the height of the other. This notion, which I express as 3 : 5 or $\frac{3}{5}$, is not a notion of the height or length of either. It is quite distinct also from the number 3 or the number 5 considered separately. It is simply the result of comparing them together, and represents the part which one is of the other, or the relation one bears to the other. Yet this relation is evidently a third magnitude, different from the two former. We should have had precisely the same result before our minds if we had compared the possessions of two persons, one of whom had £60 and the other £100. For £60 is the same part

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in the ordinary sense. The word quantity seems better fitted to describe the former of the two relations referred to in (305), and certainly does not represent Euclid's meaning. For the Greek word, πηλικότης (*pelikotes*) which he employs, is derived from πηλίκος (*pelikos*), which means, "how many times?" We have no one English word to express this, but in Latin the word "quot" was its exact equivalent. If from this word we had an English derivative "quotity," corresponding in its form to "quantity," it would express Euclid's meaning, which was—not "how much," but "how many times."

Quantus = how much ; quantity = how much-ness.

πηλίκος = quot = how many times; πηλικότης = quotity = how many times-ness. This is the Greek word which Euclid uses, and we have expressed its meaning in the text by a somewhat circuitous phrase, because the word quantity may convey an erroneous impression.

of £100 that 3 yards is of 5 yards; and as the expression 60 : 100, or $\frac{60}{100}$, represents the part which one sum of money is of the other, and the expression 3 : 5, or $\frac{3}{5}$, the part which one height is of the other, the two values 3 : 5 and 60 : 100 are equal, and the idea which they represent is precisely the same, although one is derived from the comparison of two sums of money, and the other from the comparison of two lengths.

313. Three important inferences may be deduced from these considerations:—

I. A ratio is never a concrete number, but is simply an abstract numerical notion, derived from the comparison of two magnitudes, and representing their mutual relations.

II. The two terms of a ratio are related to one another exactly as the two terms of a fraction; thus the expression $\frac{a}{b}$ means just the same as $a : b$, that is, the number of times that a contains b (182). Thus the three expressions $a \div b$, $\frac{a}{b}$, and $a : b$ are identical in meaning.

III. Every truth which may be affirmed of the numerator and denominator of a fraction applies equally to the antecedent and consequent of a ratio.

314. Hence all the propositions in (189 *et seq.*) will be found true if the words antecedent, consequent, and ratio be substituted for numerator, denominator, and fraction: *e. g.*,

If the antecedent of a ratio be increased or diminished, the ratio is increased or diminished in like manner; but if the consequent be increased the ratio is diminished, and if the consequent be diminished the ratio is increased.

Demonstrative Example.—6 : 2. Here the ratio is 3; but if we increase the antecedent 4 times it becomes 24 : 2, or 12.

General Formula.—The ratio $a : b = \frac{1}{m}$ of the ratio $ma : b$; and $a : b = m$ times the ratio $a \div m : b$.

If both antecedent and consequent be increased or diminished the same number of times the ratio remains unaltered.

Demonstrative Example.—15 : 6. Here the ratio is $\frac{15}{6}$ or $2\frac{1}{2}$; if we multiply antecedent and consequent by 4, the ratio becomes $\frac{60}{24}$, or $2\frac{1}{2}$, and if we divide both by 3 it becomes $\frac{5}{2}$ or $2\frac{1}{2}$.

General Formula.—$a : b = ma : mb = a \div m : b \div m$.

315. Proportion is the equality of ratios.

If there be any four magnitudes such that the first is as many times greater or less than the second as the third is greater or less than the fourth, these four magnitudes are said to be in proportion.

Example.—Because 5 is the same part of 15 that 20 is of 60, the four numbers, 5, 15, 20, 60, are in proportion; that is, the ratio of 5 to 15 (5 : 15) is the same as the ratio of 20 to 60. This may be thus expressed: $\frac{5}{15} = \frac{20}{60}$, or $5:15 = 20:60$, or $5:15::20:60$. The last is the ordinary method of expressing proportion, and the sign :: has the same meaning as the sign =.

The first and fourth are called the extreme terms, and the second and third the mean terms.

316. The following are examples of proportions :—

(a). 21 : 7 :: 30 : 10

(b). 7 : 9 :: 42 : 54

In (a) the common ratio is 3; that is, the number of times 21 contains 7, is $\frac{21}{7}$ or 3; so also is the number of times 30 contains 10. In (b) the common ratio is $\frac{7}{9}$. In both cases there is the same ratio between the first and second as between the third and fourth.

317. *Observation.*—Although it is necessary that the two terms of a ratio, when they are concrete numbers, should refer to the same denomination, it is not necessary that all the four terms of a proportion should be of the same name. Thus there is the same ratio between £5 and £35 as between 2 months and 14 months. Hence £5 : £35 :: 2 months : 14 months. Five pounds *are to* thirty-five pounds *as* two months *are to* fourteen months. This is a true proportion: for we do not assert equality between money and time; we only assert equality of *relationship* between two sums of money and two times; that is, the mutual relation of the two sums of money is the same as that of the two periods of time. But when numbers are once arranged in this way, it is often necessary, in practice, to disregard their special signification, and to treat them as abstract numbers.

The following is a brief investigation of the conditions under which proportion exists among abstract numbers.*

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* In Euclid, Book V., the fifth definition resembles the statement made in 318, but is somewhat more comprehensive. The reason for Euclid's wider definition is that, in geometry, proportion has often to be asserted of quantities which are incommensurable, or whose ratio cannot be numerically expressed. Thus the side of one square is to its diagonal as the side of a greater square is to its diagonal; but since the ratio of the side to the diagonal of a square is not expressible by numbers, no

318. *When four numbers are so related that any equi-multiples of the first and third can be found equal respectively to any other equi-multiples of the second and fourth, these four numbers are in proportion.*

Demonstrative Example.—4, 7, 12, 21. Here if we multiply the first and third terms by 14, we obtain 56 and 168; but on multiplying 7 and 21 by 8, we also obtain 56 and 168. Therefore 4 has the same relation to 7 as 12 to 21.

For because 14 times the first term equals 8 times the second, the first = $\frac{8}{14}$ of the second, or has to the second the ratio of 8 : 14; but in like manner, 14 times the third equals 8 times the fourth, and the third = $\frac{8}{14}$ of the fourth, or is to the fourth as 8 : 14. Wherefore the ratio of first to second = that of third to fourth.

General Formula.—If $a\ b\ c\ d$ be such numbers that $ma = nb$, and $mc = nd$, then $a : b = c : d$.

319. * *Whenever four numbers are in proportion, the product of the two extreme terms is equal to the product of the two mean terms.*

Demonstrative Example I.—This is proved by (126). For by the definition of proportion, if the first be greater than the second, the fourth is exactly as many times less than the third; and if the first be less than the second, the fourth is as many times greater than the third. Wherefore the product of the second and third must equal that of the first and fourth.

Demonstrative Example II.—In the proportion 48 : 16 :: 27 : 9, let us find the common ratio 3, and multiply the two consequents by it, then the numbers become 48 : 48 :: 27 : 27. But as one of the extremes and one of the means have been multiplied by the same number, the product of the extremes and the product of the means have been also multiplied by the same number. But it is evident that in the second case the products are equal; therefore they were equal before both were multiplied by the same number.

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multiple of a side can ever be found equal to a multiple of a diagonal. Yet there certainly exists a proportion or equality of ratios between the first two magnitudes and the second. Euclid's definition is intended to meet such cases; but in Arithmetic every magnitude is numerically expressed, and therefore the statement given in the text is sufficient, and is invariably true.

* Euclid, book vii., proposition xiv.

General Formula.—If $a : b :: c : d$, then $ad = bc$;

For (313) if $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$.

Multiply both by bd, then $\frac{abd}{b} = \frac{cbd}{d}$

Cancel the common factors, then $ad = cb$.

320. TO FIND ANY TERM OF A PROPORTION IF THE REMAINING THREE ARE KNOWN—

RULE.

If the required be a mean term, find the product of the extremes and divide by the remaining mean; if the required term be an extreme, find the product of the mean terms, and divide by the extreme.

For if $a : b :: c : d$, then (319) $ad = cb$.

Hence (125) $a = \frac{cb}{d}$, $d = \frac{cb}{a}$, $c = \frac{ad}{b}$, and $b = \frac{ad}{c}$.

Example.—What is the third term of a proportion of which the first is 150, the second 80, and the fourth 12?

Here, because 150 : 80 :: the third term : 12;

By (125) the third term $= \frac{150 \times 12}{80} = 22{\cdot}5$.

EXERCISE XCIX.

☞ Fill up the vacant places in the following proportions:—

1, 7 : 42 :: ( ) : 120.   2. 28 : ( ) :: 100 : 25.
3. ( ) : 17 :: 804 : 67.   4. 12 : 58 :: 342 : ( )
5. ( ) : 79 :: 683 : 15.   6. 27 : ( ) :: 108 : 97.
7. 832 : 122 :: ( ) : 2440.   8. ( ) : 50 :: 9 : 1.
9. 264 : 1000 :: 66 : ( ).   10. 287 : 372 :: ( ) : 5376.

321. *Whenever the product of two extreme terms equals that of the two mean terms, the four numbers so arranged are in proportion.*

Demonstrative Example.—Because 3 × 12 = 4 × 9.

Therefore 3 : 4 :: 9 : 12.

General Formula.—Let $ab = cd$, divide both by cb, $\frac{ab}{cb} = \frac{cd}{cb}$.

Reduce them to the lowest name, $\frac{a}{c} = \frac{d}{b}$, or $a : c = d : b$.

322. *Corollary.—Whenever the product of two numbers equals the product of two others, and the four numbers are so arranged that one pair of factors forms the two extremes, and the other pair the means, the numbers thus arranged shall be in proportion.*

If 6 × 12 = 9 × 8,	If $ad = cb$,
Then 6 : 9 :: 8 : 12.	$a : b :: c : d$
12 : 9 :: 8 : 6	$c : a :: d : b$
9 : 6 :: 12 : 8	$d : c :: b : a$
6 : 8 :: 9 : 12.	$a : c :: b : d$

EXERCISE C.

☞ Arrange the terms of each of the proportions in the last Exercise in as many ways as you can, and make six other examples.

323. *If two numbers be prime to each other, no smaller numbers can be found which have the same ratio.*

324. *Demonstrative Example.*—If 4 and 9 are prime to one another, the ratio 4 : 9, or $\frac{4}{9}$, cannot be expressed by any smaller numbers than 4 : 9. For if it were possible, then both antecedent and consequent could be divided by some one number, and the two quotients would represent their ratio. But this number would necessarily be a common divisor of 4 and 9. Whereas by the hypothesis they have no common measure.*

325. It is often convenient to express a given ratio in its lowest terms, and this is done by dividing antecedent and consequent by their greatest common measure.

EXERCISE CI.

☞ (a). Express the following ratios in their lowest terms:—

1. 14 : 77; 200 : 384; 616 : 7056; 63 : 135.
2. 936 : 2368; 81 : 4872; 220 : 528; 35 : 315.
3. 4067 : 2573; 856 : 936; 1242 : 2323; 176 : 2000.

(b). Make twelve different proportions, and show—1. That equi-multiples of the first and third can be found equal to other equi-multiples of the second and fourth. 2. That the product of the first and fourth equals that of the second and third; and 3. That each proportion may be stated in four different ways.

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* Euclid, Book vii. prop. 23.

** SECTION II.

The following propositions are true in all cases of proportion :—

325. *I.—If four numbers be in proportion, the sum or difference of the first and second is to the second, as the sum or difference of the third and fourth is to the fourth.*

If $20 : 12 :: 5 : 3$; then $20 \pm 12 : 12 :: 5 \pm 3 : 3$.

Let $w : x :: y : z$; then (313) $\frac{w}{x} = \frac{y}{z}$. Add 1 to both equals;

then $\frac{w}{x} + 1 = \frac{y}{z} + 1$. But $1 = \frac{x}{x}$ or $\frac{z}{z}$ Therefore $\frac{w}{x} + \frac{x}{x} = \frac{y}{z} + \frac{z}{z}$

or $\frac{w + x}{x} = \frac{y + z}{z}$. Hence $w + x : x :: y + z : z$.

Again, subtract 1 from both sides of $\frac{w}{x} = \frac{y}{z}$. Then $\frac{w}{x} - \frac{x}{x} = \frac{y}{z} - \frac{z}{z}$,

or $\frac{w - x}{x} = \frac{y - z}{z}$, or $w - x : x :: y - z : z$.

Hence $w \pm x : x :: y \pm z : z$.

327. *II.—If there be two equal ratios, the sum or difference of the antecedents is to the sum or difference of the consequents, as either of the antecedents is to its consequent.*

If $100 : 25 :: 16 : 4$, then $100 \pm 16 : 25 \pm 4 :: 100 : 25 :: 16 : 4$.

Let $w : x :: y : z$; then $\frac{w}{x} = \frac{y}{z}$ Also by (322) $\frac{w}{y} = \frac{x}{z}$. But

(326) if $\frac{w}{y} = \frac{x}{z}$, then $\frac{w \pm y}{y} = \frac{x \pm z}{z}$, or $w \pm y : y :: x \pm z : z$

Hence by (322) $w \pm y : x \pm z :: y : z :: w : x$.

328. *Corollary.*—The sum of the antecedents is therefore to the difference of the antecedents, as the sum of the consequents is to the difference of the consequents. For these ratios have each been proved equal to the original ratio. Hence they are equal to one another.

329. *III.—If there be any number of equal ratios, the sum of all the antecedents is to the sum of all the consequents, as either of the antecedents is to its consequent.**

$$3 : 5 :: 9 : 15 :: 18 : 30 :: 330 : 550.$$

Then $\frac{3 + 9 + 18 + 330}{5 + 15 + 30 + 550}$ $\frac{3}{5} = \frac{9}{15} = \frac{18}{30} = \frac{330}{550}$

Euclid, Book vii. 12.

Let $a:b::c:d::e:f$, then $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{c}{a} = \frac{e}{f}$. For by (327) if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+c}{b+d} = \frac{a}{b}$. But $\frac{}{b} = \frac{e}{f}$. Wherefore $a+c:b+d::e:f$. The sum of these antecedents and consequents may again be taken, when $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{e}{f}$.

330. *IV.—If any number of pairs of equal ratios be taken and arranged in like order, and all the antecedents on each side be multiplied together for new antecedents, and also all the consequents on each side for new consequents, the four products thus obtained will also be in proportion.*

If $4:7::12:21$; $5:18::120:432$; and $13:9::39:27$; then $4 \times 5 \times 13 : 7 \times 18 \times 9 :: 12 \times 120 \times 39 : 21 \times 432 \times 27$.

If $a:b::c:d$
and $l:m::n:r$
and $w:x::y:z$

Then $alw:bmx$
$=cny:drz.$

For if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} \times \frac{l}{m} = \frac{c}{d} \times \frac{n}{r}$, because by hypothesis $\frac{l}{m} = \frac{n}{r}$; and we have therefore only multiplied two equals by the same ratio. So because $\frac{al}{bm} = \frac{cn}{dr}$, $\frac{al}{bm} \times \frac{w}{x} = \frac{cn}{dr} \times \frac{y}{z}$.

331. *Corollary.*—If four numbers are in proportion their squares are in proportion, also their cubes, their fourth powers, &c.

For it is evident that if the proportion $a:b::c:d$ be compounded with itself by multiplication, as described in (330), the result will be

$$\frac{a^2}{b^2} = \frac{c^2}{d^2}, \text{ or } \frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

By means of that alternation of the order of the several terms, which is indicated in (322), the results of the four last propositions may be varied in several ways.

Exercise CII.

☞ Verify each of the last four propositions by the proportions given in Exercise XCIX., and by twelve other proportions of your own selection.

Example.—Because 8 : 5 :: 24 : 15.
By 326. 13 : 5 :: 39 : 15, and 3 : 5 :: 9 : 15.
By 327. 32 : 20 :: 8 : 5, and 16 : 10 :: 24 : 15.
By 328. 13 : 3 :: 39 : 9.
By 331. 64 : 25 :: 576 : 225.

SECTION III.—THE RULE OF THREE, OR SIMPLE PROPORTION.

332. The rule by which the principles of Proportion are applied to practice is called the Rule of Three.

It is so called because in all the questions proposed in it *three* terms of a proportion are given, and it is required to find a fourth.

333. *Observation.*—The doctrine of proportion affects more or less every part of arithmetic. In many of the processes we have already employed, the existence of proportion among the several numbers has been assumed. Thus in Compound Multiplication, if we ask, What will be the price of 79 articles if one costs 8s. 6d.? it is assumed that there is to be the same ratio between the number of articles as between the prices paid for them. So when in Division the question occurs, "If 20 men dig 45 acres of ground, what will one dig?" it is assumed that the quantity of work done will be proportioned to the number of workers; and we are asked to find a number of acres bearing the same ratio to 45 that 1 bears to 20.

334. In every sum in Simple Proportion two things are required:

I. To determine in what relation the unknown number is to stand to the given number of the same denomination.

II. To find a number which shall fulfil the required conditions.

335. The former of these is in fact a *theorem* in Proportion, and requires to be thought out by the help of the principles laid down on that subject. The latter is merely a *problem* in Multiplication and Division. For when three terms of a proportion are known and arranged, the fourth may be readily found by (320), because in all such cases the product of the extremes equals the product of the means. But before that rule can be employed it is necessary to arrange the terms, and to do this we must first discover in what relation the required answer must stand to a certain known quantity, and then lay out the given terms in such a way as to represent that relation. This task of *arranging the terms* forms the principal difficulty of the Rule of Three.

336. *Direct Proportion.*—When any two magnitudes are so related that as the one increases or diminishes the other increases or diminishes, they are said to vary *directly*, or to be in *Direct Proportion* the one to the other.

Example.—If a number of men are employed upon a certain work, the more men are hired the more work will be done, and the less the work to be done the less men will be required. Hence the number of men and the quantity of work are in *direct proportion*, or the one number *varies directly* as the other; for to increase or diminish the one is to increase or diminish the other.

337. In the same way it appears that, all other things being equal—

The price of any number of articles varies as the number of articles.
The cost of conveyance varies as the distance travelled.
The rent of land varies as the time of the tenure.
The wages of a labourer vary as the time of his employment.
The interest paid for the use of money varies as the sum lent.
The taxes paid by a householder vary as the rental of his house.
Work done in a given time varies as the number of agents.

For let a and b be any numbers; then—

The price of a articles : the price of b articles :: $a : b$.

The cost of carrying luggage a miles : the cost of carrying it b miles :: $a : b$.

The rent of land for a years : the rent for b years :: $a : b$.
Labourers' wages for a weeks : wages for b weeks :: $a : b$.
Interest paid upon £a : interest paid on £b :: $a : b$.
Taxes paid on a rental a : taxes paid on a rental b :: $a : b$.
Work done by a men , work done by b men :: $a : b$.

Observation.—In every one of these cases there is the same ratio between two magnitudes of one kind as between the two corresponding terms of another kind. The term relating to the number a is always in the first place and forms the antecedent of a ratio, while the number a itself is always in the third term and forms the antecedent of another ratio. Thus in every case the two antecedents belong to one another in the same way as the two consequents, and the proportion is said to be *direct*.

338. *Inverse Proportion.*—When any two magniudes are so related that as the one increases the other diminishes, or as the one diminishes the other increases, they are said to vary *inversely*, or to be in Inverse or *Indirect Proportion* the one to the other.

Example.—If a given work has to be performed and a number of men is hired to do it, the greater the number of men the less time will be occupied, and the greater the time allowed the fewer men will be required. Here the number of men required is in *inverse* proportion to the time employed; to increase the one is to diminish the other, and *vice versâ.*

339. In like manner it appears that, all other things being equal—

Length of pieces (of carpet, &c.) required to cover a surface varies inversely as the breadth of the stuff.

Number of coins required to pay a certain sum varies inversely as the value of the coins.

Velocity of a body moving through a space varies inversely as the time occupied in its passage.

Weight of luggage carried for a given sum varies inversely as the distance of conveyance.

For let a and b again represent any numbers; then—

Length of carpet a feet wide required to cover a certain floor : length of carpet b feet wide to cover the same :: b : a.

Number of coins value a required to pay a certain sum : number value b to pay the same :: b : a.

Velocity of a body traversing a certain distance in a seconds : velocity of a body traversing the same distance in b seconds :: b : a.

Weight of luggage carried a miles for a certain sum : weight carried b miles for the same sum :: b : a.

Observation.—In every one of these cases the increase of one term in a ratio makes it necessary to diminish the corresponding term in the other ratio, and the two kinds of number are said to be *reciprocally* or *inversely* proportional the one to the other, the terms of the second ratio being *inverted* with respect to the first. Here it will be seen that if the term belonging to the number a is in the first place and forms the antecedent of one ratio, the number a itself is in the *fourth* place and forms the consequent of the other ratio.

The meaning of direct and inverse proportion becomes further evident when we consider the nature of a fraction, thus—

340. *If two fractions have the same denominators, they are to one another* as *their numerators*, i.e., *they varv directly as their numerators.*

Demonstrative Example.— $\frac{5}{16} : \frac{7}{16} :: 5 : 7$.

For it has already been shown (189) that fractions having the same denominators are greater or less according as their numerators are greater or less, and it is obvious that the product of the first and fourth ($\frac{5}{16} \times 7$) equals the product of the second and third ($\frac{7}{16} \times 5$). Hence the four terms make a proportion (321).

The fraction $\frac{5}{16}$ and the 5 which corresponds to it are the two antecedents, and $\frac{7}{16}$ and 7 are the two consequents. The proportion is therefore *direct.*

General Formula.—$\frac{a}{x} : \frac{b}{x} :: a : b$; for $\frac{a}{x} \times b = \frac{b}{x} \times a$.

341. *But if two fractions have the same numerator but different denominators, they are to one another in the inverse ratio of their denominators*, i.e., *they vary inversely as their denominators.*

Demonstrative Example.— $\frac{7}{9} : \frac{7}{12} :: 12 : 9 = \frac{1}{9} : \frac{1}{12}$.

For it has already been shown (190) that with the same numerator a fraction is greater as its denominator is less, and less as its denominator is greater. It is also true that $\frac{7}{9} \times 9 = \frac{7}{12} \times 12$. Wherefore (322) fractions vary inversely as their denominators.

General Formula.— $\frac{a}{x} : \frac{a}{y} :: y : x = \frac{1}{x} : \frac{1}{y}$.

342. If we bear in mind the connexion between the formulæ of Fractions and those of Division, the last statements will appear to be equivalent to the following :—

If two Division sums have the same divisor but different dividends, the quotients vary directly as their dividends.

But if two Division sums have the same dividends but different divisors, the quotients vary inversely as their divisors.

343. The reasoning required in every Rule of Three sum takes the following form. We must—

I. Ascertain of what kind or denomination the answer is to be.

II. Select that one of the three given terms which relates to the same kind as the answer.

III. Make this term and the unknown term the antecedent and consequent of a ratio.

IV. Make the remaining terms the antecedent and consequent of another ratio, arranged so that the proportion may be true.

344. When the terms have been thus arranged, the solution of the question depends on the principle already explained (319), and on the rule which is founded on it, viz.:—

Take the product of any pair, either of extreme or mean terms, and divide it by the remaining one; the quotient will be the fourth proportional required.

Observation.—The denomination of the number thus found is always the same as that which forms its antecedent or consequent.

345. *Example* I.—Find the cost of 113 lbs. when 7 lbs. cost 20 pence.

Here the price of a certain weight is given and it is required to find the price of another weight. But because price is directly proportioned to the quantity purchased, the two weights must have the same ratio as the two sums of money.

Hence let x equal the unknown sum of money, then—

$$\begin{array}{rcrcrcr} 20\text{d.} & : & x\text{d.} & :: & 7\text{ lbs.} & : & 113\text{ lbs.} \\ 113\text{ lbs.} & : & 7\text{ lbs.} & :: & x & : & 20. \\ x & : & 20 & :: & 113 & : & 7. \\ 7 & : & 113 & :: & 20 & : & x. \end{array}$$

Each of these proportions must be true; for in all, the two weights form the antecedent and consequent of one ratio, and the two prices (*i.e.*, the known and the unknown sum of money) form the corresponding antecedent and consequent of the other ratio.

In each of these cases, therefore, the term required is to be found by multiplying 113 by 20 and dividing by 7, or

$$x=\frac{113\times 20}{7}=\overset{d.}{322.85}=\overset{£.}{1}\ \ \overset{s.}{6}\ \ \overset{d.}{10.85}.$$

Observation.—It was not necessary to consider any of the numbers as concrete while the sum was being worked, for all that was required was to find a number which should bear the same relation to 7 as 113 did to 20. But this number when found is of the same denomination as the 7; *i.e.*, it is a number of pence, and the answer has to be reduced to pounds.

346. *Example II.*—25 men reap 60 acres in a certain time, how many acres will 4 men reap in the same time?

Here we want to find a certain number of acres, let it be x.

But 60 is a given number of acres, and these two numbers have a ratio one to the other—$60 : x$.

But (337) work done is directly proportioned to the number of agents.

Therefore as the number of men who reap 60 acres is to the number who reap x, so is 60 to x.

$$\begin{aligned} \text{Hence } 25 &: 4 :: 60 : x. \\ \text{Or } 60 &: x :: 25 : 4. \\ \text{Or } x &: 60 :: 4 : 25. \\ \text{Or } 4 &: 25 :: x : 60. \end{aligned}$$

We want a number of acres which shall be as many times less than 60 as 4 is less than 25, and either of these proportions will give us such a number; for—

$$x = \frac{4 \times 60}{25} = 9{\cdot}6 = 9{\cdot}6 \text{ acres.}$$

347. *Example III.*—If 120 yards of carpet 3 qrs. wide will cover a floor, how many yards would be required if the carpet were 5 qrs. wide?

Here the answer (x) will require to be the length of carpet.

But 120 is the given term which represents length of carpet.

$\therefore$ $120 : x$ is the ratio to be solved.

But the wider the carpet is the less length will be required, for (339) with a given surface length and breadth are *inversely* proportioned to each other.

Hence x must be less than 120 as many times as 5 is greater than 3.

$$\begin{aligned} x &: 120 :: 3 : 5. \\ \text{Or } 3 &: 5 :: x : 120. \\ \text{Or } 120 &: x :: 5 : 3. \\ \text{Or } 5 &: 3 :: 120 : x. \end{aligned}$$

In every case $\frac{120 \times 3}{5} = 72$ yards, which is the answer required.

348. From (311) it appears that if two numbers are not of the same denomination they do not represent the ratio of the magnitudes which they express. Hence if the two terms of any ratio are not in the same name they must be made so before the sum can be worked. This can always be done by reducing the numbers of the higher into those of the lower denomination.

349. *Example IV.*—If 2 tons 9 cwt. cost £114 16s., what will 26 lbs. cost?

Here a sum of money is the answer required; let it be x.

But £114 16s. is a sum of money.

∴ £114 16s. : x is the ratio required.

But (337) the quantity purchased is *directly* proportioned to the sum spent.

Hence, as the price of 2 tons 9 cwt. is to the price of 26 lbs., so is 2 tons 9 cwt. to 26 lbs., or £114·8 : x :: 2 tons 9 cwt. : 26 lbs. But the figures in this statement do not represent the ratio properly because they are concrete numbers of different denominations.

Wherefore reducing 2 tons 9 cwt. to pounds, and £114 16s. to the decimal of £1—

$$\begin{array}{rcrcrcl} x & : & 114{\cdot}8 & :: & 26 & : & 5488. \\ 114{\cdot}8 & : & x & :: & 5488 & : & 26. \\ 26 & : & 5488 & :: & x & : & 114{\cdot}8. \\ 5488 & : & 26 & :: & 114{\cdot}8 & : & x. \end{array}$$

In each case the answer will be found in the same manner.

$$x = \frac{26 \times 114{\cdot}8}{5488} = \overset{\pounds}{{\cdot}543} = \overset{s.}{10}\ \overset{d.}{10\tfrac{1}{2}}$$

350. Sometimes a number is mentioned in the question which does not affect the answer, and need not be stated.

Example V.—If I can have 3 cwt. 2 qrs. carried 80 miles for 16s., how far can I have 2 tons carried for the same money?

Here the 16s. has nothing to do with the question, for if any other sum were mentioned the answer would be just the same.

The answer is to be a certain distance.

∴ 80 : x is the ratio to be determined.

But (339) the distance is inversely proportioned to the weight; for the greater the weight the less distance it will be carried for a given sum.

Hence the answer will be less than 80 miles, and the consequent of the first ratio must be less than its antecedent.

∴ 2 tons : 3 cwt. 2 qrs. :: 80 : x.

Reducing the first two terms to the same name—

160 qrs. : 14 qrs. :: 80 : x.
Or 80 : x :: 160 : 14.
14 : 160 :: x : 80.
x : 80 :: 14 : 160.

In all the cases—

$$x = \frac{80 \times 14}{160} = 7 \text{ miles.}$$

351. *Example VI.*—A father divides £1280 among his three children, so that their portions are as 5, 3, and 2 respectively; how much does each receive?

Here are three several sums to be worked, as we want three answers. We first notice that, as the answers will be in money, £1280 : x is the ratio to be determined in each case.

Now 5 + 3 + 2 = 10. Wherefore the first is to have £5 out of every £10, the second £3, and the third £2.

∴ 10 : 5 :: 1280 : Share of the first
And 10 : 3 :: 1280 : Share of the second.
And 10 : 2 :: 1280 : Share of the third.

∴ the three answers are £640, £384, and £256, respectively.

352. *Example VII.*—£4000 have to be divided among 4 persons in the proportions of $\frac{1}{6}$, $\frac{2}{5}$, $\frac{4}{9}$, $\frac{1}{3}$; what is the share of each?

Reducing these fractions to a common denominator, we find them $\frac{15}{90}$, $\frac{36}{90}$, $\frac{40}{90}$, $\frac{30}{90}$, in all $\frac{121}{90}$. The 90 does not affect the question, but the whole must be divided into 121 parts.

∴ 121 : 15 :: 4000 : Share of the first.
121 : 36 :: 4000 : Share of the second.
121 : 40 :: 4000 : Share of the third.
121 : 30 :: 4000 : Share of the fourth.

The four answers will be found to make £4000 if added together, and to be to one another in the proportion of $\frac{1}{6}$, $\frac{2}{5}$, $\frac{4}{9}$, and $\frac{1}{3}$.

353. *Observation.*—In the examples just given we have placed the symbol representing the required answer, sometimes in one place of the proportion and sometimes in another. It is important to notice that any one of these places (first, second, third, or fourth) is just as suitable as any other, provided the remaining terms be so placed as to form a true proportion. Nevertheless, it is usual *to reserve the fourth place for the answer in all cases*, and to arrange the remaining terms according to the following rule :—

RULE OF THREE.

354. Place the term which is of the same kind as the required answer in the third place, leaving the fourth place for the answer.

If the answer will be greater than the third term, make the second term greater than the first; but if the fourth is to be less than than the third, then place a less term in the second place than in the first.

If the first and second terms be not of the same denomination reduce the higher to the same name as the lower (84).

Then multiply the second and third terms together, and divide by the first.

The quotient thus found is of the same name as that to which the third term has been reduced.

EXERCISE CIII.

1. How many yards of cloth, worth 18s. 3½d. per yard, must be given in exchange for 937½ yards, worth 3s. 7½d. per yard?

2. If an insolvent, whose whole possessions amount to £1,728, pays at the rate of 4s. 3d. in the pound, what does he owe?

3. If an insolvent, whose debts amount to £2,740, pays £67 to a creditor whose account is £162 15s. [illegible]t is the total value of his effects?

4. If £1,250 will purchase 2½ tons, what weight can be bought for £2 3s. 9d.?

5. How many coins worth 2s. 3d. each are equal to 10,000 German kreutzers, three of which are worth an English penny?

6. Two numbers together make 1,800, and they are to one another as 2 to 7; what are they?

7. A man divides £17 2s. 10d. among four people, so that for every 10d. one has, another has 9d., another 8d., and the fourth, 7d.; how much did each receive?

8. The ratio of the diagonal to the side of a square is about as 99 to 70; what is the diagonal of a square whose side is 3·5 miles? and what is the side of a square whose diagonal is 172·3 yards?

9. If $\frac{5}{7}$ of a pound cost $\frac{4}{9}$ of a shilling, what will $\frac{3}{8}$ of a cwt. cost?

10. If 7 hogsheads, each weighing 3 cwt. 2 qrs. 17 lbs., can be purchased for £54, what must the goods be sold at per lb. to gain £15 on the whole transaction?

11. A bankrupt owes £7,250, and his entire property amounts to £587 10s.; what dividend can he pay?

12. Find a fourth proportional to ·0004, 1·4, and ·02; and also to $\frac{2}{7}$, $\frac{3}{4}$, and $\frac{5}{8}$.

13. Find the difference between the cost of 37 cwt. 2 qrs. 14 lbs. at £7 10s. 9d. per cwt., and that of 39 cwt. 3 qrs. 26 lbs. at £4 17s. 10d. per cwt.

14. If a man's wages are regulated by the price of provisions, and he receives 1s. 3d. a day when corn is 7s. per bushel, what will he receive when corn is 5s. 6d. per bushel?

15. If a piece of cloth measuring 9 ells 1 nail 1·125 inches cost £4 10s. 6½d., what is the price per yard?

16. Two couriers pass through a town at an interval of 4 hours, travelling at the rate of 11½ and 17½ miles an hour; how far and how long must the first travel before he is overtaken by the second?

17. Gravity varies directly as the mass and inversely as the square of the distance. Compare the amount of the earth's attraction on two bodies, the one having a mass 35 at a distance 6, and the other having a mass 125 at a distance 11.

18. How many yards of lace can I buy for £228 12s. 7d., at the rate of £1 15s. for 4¼ yards?

19. If I buy 7½ dozen of champagne at £3 5s. per dozen, and 13 dozen 7 bottles of port at 42s. per dozen, what is the amount of my bill?

20. If 2 cwt. 1 qr. 17 lbs. cost £50 11s. 4d., how much can be bought for £18 7s. 6d.?

21. If a street three-quarters of a mile long be repaired at a cost of £7 9s. 6d., what portion of the expense should be paid by an inhabitant whose premises have a frontage of 18 yards 2 feet?

22. How many reams of paper could be bought for £120, at the rate of £15 6s. for 13 reams.

23. If a train runs from London to Sydenham, a distance of 5½ miles, in 12 minutes 5 seconds, how long should it take to reach a place which is 49 miles 3 furlongs distant?

24. The areas of circles are as the squares of their diameters. Suppose there are two circles whose diameters are as 11 to 17·5, and the

area of the first contains 3 acres 2 roods 17 poles; what is the area of the second?

25. If a tradesman who owes £1,850 is only worth £567 10s., what ought he to pay in the pound? and how much will a creditor receive whose claim is £58 15s.?

26. If a tradesman, whose debts amount to £2,653 10s. 6d., pays 6s. 3½d. in the pound, what is the worth of his effects?

27. If a bankrupt, whose whole possessions amount only to £625 13s., pays a dividend of 1s. 3¼d. in the pound, what does he owe?

28. If I lend £112 for 9 months, how much should the borrower lend me for 6 months, in order to repay the loan?

29. Find the fourth term of the following proportion :—

$$40{\cdot}5 : 3^2 + 5^2 - 2 :: 3^5 \div 3^2 : (\quad).$$

30. If 49 yards of carpeting 34 inches wide cover a floor, how much will be needed of carpet which is only 29 inches broad.

31. The French "aune" is to the English yard as 1000 to 769. What is the length in French measures of 123 yards 2 feet?

32. When the annual value of rateable property in the county of Middlesex was returned at £7,584,668, what did a tax of 2¾d. in the pound produce if levied on the whole?

33. Find the cost of 39 cwt. 3 qrs. 26 lbs., at £4 17s. 10d. for 2 cwt. 1 qr. 19 lbs.

34. If a tower 45 feet high has a shadow of 39 feet, what is the height of an obelisk having at the same time a shadow 76·5 feet long?

35. If a servant whose yearly wages are 25 guineas enters service on the 27th of August, and quits it on the 13th of the following January, what ought he to receive?

36. If, after paying income tax at 1s. 2d. in the pound, a gentleman has £701 10s. 10d. remaining, what is his annual income?

37. If 12.35 lbs. are sold for 7·4 shillings, what quantity could I buy for £253·67?

38. A parish, whose rental is £6,870, requires for the support of the poor £287 per annum; what should be the rate per pound?

39. If 40 men do a piece of work in 27 days, how many men will do 8 times as much in 135 days?

EXERCISE CIV.

☞ State each of the last six sums in four ways, making the answer (x) in the first, the second, the third, and the fourth terms of the proportion, as in the examples.

SECTION IV.—COMPOUND PROPORTION, OR THE DOUBLE RULE OF THREE.

355. When the antecedents of two or more ratios are multiplied together and also their consequents, the ratio thus produced is said to be *compounded* of the others.

Thus if we take the two ratios, 7 : 9 and 3 : 4, and multiply their antecedents and consequents so as to get 21 : 36, this is the ratio compounded of 7 to 9 and of 3 to 4.

356. Composition of ratios is exactly equivalent to the multiplication of fractions.

$$\text{Thus } \frac{5}{6} \text{ of } \frac{7}{12} = \frac{35}{72}, \text{ or } \begin{array}{l} \left\{\begin{array}{r} 5 : 6 \\ 7 : 12 \end{array}\right. \\ \hline 35 : 72 \end{array}$$

357. In solving many sums in Proportion it is necessary to compound one ratio with another in this way. Whenever this is done the sum is said to be in Compound Proportion, or the Double Rule of Three.

The employment of this Rule depends on the following principle:—

358. *If there be three quantities so related that when all other things are equal the first varies as the second, and also when all other things are equal the first varies as the third; the variation of the first is as the product of the second and third.*

Demonstrative Example.—If work done varies as the time of doing it, and also as the number of persons employed; it varies altogether as the product of the number of days, and the number of workers.

For if a certain work is done by one man in one day, 3 men in 4 days will do 3×4, or 12 times as much.

Here the work done is affected by two circumstances—the time, and the number of labourers. But as the time is increased four times, this circumstance increases the work four times. Besides this, the number of men is increased three times. Therefore, as one circumstance tends to increase the answer in the ratio of 3 to 1, and the other in that of 4 to 1, the total increase arising from both is 3×4: 1, or as 12 to 1.

General Formula.—If a increases as b increases, and also as c increases, then the total increase of a is as bc.

359. Whenever two ratios separately affect a third, their total effect upon it is represented by the ratio compounded of those two, *i. e.*, by the ratio formed by the product of the two antecedents and the product of the two consequents.

360. *Example.*—One ball weighing 5 pounds, and moving at the rate of 20 feet per second, strikes a wall with a certain force; what is the relative force of the blow given by another ball of 11 pounds weight, and moving 30 feet per second?

Now here the second ball weighs more than the first in the ratio of 11 to 5; ∴ as far as the weight is concerned the blow of the second ball will be to that of the first as 11 to 5, or 11 : 5, or $\frac{11}{5}$.

But the second also moves more quickly; ∴ as far as the rapidity is concerned, the blow of the second will be greater than the first in the ratio of 30 to 20, *i. e.*, 30 : 20, or $\frac{30}{20}$.

Then, because one circumstance tends to increase the answer in the proportion of 11 to 5, and another tends to increase it in the proportion of 30 : 20; the two together will increase it as $\frac{30}{20} \times \frac{11}{5}$, or $\frac{330}{100}$. Therefore the blow in the second case will be 3·3 times that in the first.

361. *Observation.*—The general numerical formula for momentum, when the velocity and the weight can both be expressed in figures, is Momentum = Weight × Velocity.

RULE FOR COMPOUND PROPORTION.

362. Find the number which is of the same kind as the required answer, and place it in the third term, as in Simple Proportion.

Take any two terms which are of the same kind, and consider these two with the third as a separate sum in Simple Proportion. State it according to (354).

Then take the other two which are of the same kind, and treat them, with the third term, as another distinct sum in Simple Proportion. State it also according to the same Rule.

Arrange all the separate ratios the one beneath the other; then multiply all the first terms together for a new first term or antecedent, and all the second terms together for a new second term or consequent.

Then multiply this new second term by the third and divide by the first, as in Simple Proportion.

363. *Example I.*—If 13 men earn £7 in 6 days, what will be the wages of 72 men for 21 days?

This sum can be broken into two, thus :—

(I.) If 13 men earn £7, what will 72 men earn?

By Rule (354) this will be stated—13 : 72 :: 7 : x.

(II.) If £7 are earned in 6 days, what will be earned in 21 days?

And by Rule (354) this will be stated—6 : 21 :: £7 : x.

The two statements must be thus combined :

$$\begin{array}{lrcr} \text{Men} & 13 & : & 72 \\ \text{Days} & 6 & : & 21 \end{array} \Bigg\} :: \overset{£}{7} : \overset{£}{x}$$

$$\overline{78 : 1512} \quad :: 7 : x$$

And the answer is $\frac{1512 \times 7}{78}$ = £135·692 = £135 13s. 10¼d.

364. *Observation I.*—While we considered the third term in relation to the number of labourers, we neglected the consideration of the time; and while the third term and the time were considered, we neglected to take into account the number of men. Each question had to be determined on its own merits as a separate sum in Simple Proportion; and then the two ratios were compounded.

365. *Observation II.*—Here are two conditions affecting the amount of the wages to be paid, and each of them tends to increase the answer. First, there are more men to pay; and this would make it necessary to have an amount greater than £7, as much as 72 is greater than 13.

Hence as far as the number of labourers is concerned we want an amount which shall be to £7 as 72 is to 13; *i. e.*, we want $\frac{72}{13}$ of £7.

But the time of their engagement is also greater.

Therefore as far as the time is concerned we want an amount as much greater than £7 as 21 is greater than 6; *i.e.*, we want $\frac{21}{6}$ of £7.

Hence the answer required is neither of these fractions separately, but the combined effect of the two, and we find this by multiplying the two fractions, or compounding the two ratios.

For the fourth term is to be, not $\frac{72}{13}$ of £7 only, nor $\frac{21}{6}$ of £7 only, but $\frac{72}{13} \times \frac{21}{6}$ of £7. And the ratio required is

$$13 \times 6 : 72 \times 21 :: 7 : x$$

366. *Example II.*—If in a certain business £900 profit be realized on an investment of £5,200 for 10 months, what should be the profit accruing on £7,950 invested for six months?

Here the answer is to be *profit;* and £900 represents profit. ∴ £900 : x is the ratio to be solved.

There are two separate sums here.

I. If a man with a capital of £5,200 realizes £900, what should he realize whose capital is £7,950?

By Rule (354) this must be stated—5,200 : 7,950 :: 900 : x.

II. If £900 are realized in 10 months, what should be realized in 6?

By Rule (354) 10 : 6 :: 900 : x.

Combining these two statements—

$$\left.\begin{matrix} 5200 : 7950 \\ 10 : 6 \end{matrix}\right\} \quad :: 900 : x$$

$$\overline{5200 \times 10 : 7950 \times 6 :: 900 : x}$$

$$\therefore x = \frac{7950 \times 9 \times 900}{5200 \times 10};$$

or by cancelling, $x = \dfrac{795 \times 6 \times 9}{52} = £825{\cdot}576 = £825$ 11s. $6\frac{6}{25}$d.

367. *Example III.*—If 12 horses plough a field of 8 acres in 3 days, in what time will 21 horses plough a field of 100 acres?

Here because the answer is to be time, 3 days is the third term, and the ratio to be determined is 3 : x.

There are here two sums.

I. If in 3 days 12 horses do any work, in what time will 21 horses do it?

This is a case of Inverse Proportion (338).

Therefore the statement is 21 : 12 :: 3 : x.

II. If in 3 days 8 acres are ploughed, in what time will 100 acres be ploughed?

This is a case of Direct Proportion (336).

Therefore the statement is 8 : 100 :: 3 : x.

Combining these two statements—

$$\left.\begin{matrix} 21 : 12 \\ 8 : 100 \end{matrix}\right\} \quad :: 3 : x$$

$$\overline{21 \times 8 : 100 \times 12 :: 3 : x}$$

$$x = \frac{100 \times 12 \times 3}{21 \times 8} = 21{\cdot}428 \text{ days.}$$

368. *Example IV.*—If 125 men can dig a trench 100 yards long, 20 yards wide, and 4 feet deep, in 4 days, working 12 hours a day; how many men must be engaged to dig another trench 500 yards long, 8 yards wide, and 6 feet deep, in 3 days, working $7\frac{1}{2}$ hours per day?

Here the answer is to be a number of men, and 125 is of the sam kind, wherefore 125 : x is the ratio to be solved.

Several conditions affect this ratio, and each may be made in turn the subject of a Simple Proportion sum.

I. If 125 men dig a trench 100 yards long, how many will dig one 500 yards long?

By Rule (354) the statement is 100 : 500 :: 125 : x.

II. If 125 men dig a trench 20 yards wide, how many will be required to dig one 8 yards wide?

Here the statement is 20 : 8 :: 125 : x.

III. If 125 men dig a trench 4 feet deep, how many will dig one 6 feet deep?

By Rule (354) the statement is 4 : 6 :: 125 : x.

IV. If 125 men do a certain work in 4 days, how many will do it in 3?

The proportion being inverse, this statement is 3 : 4 :: 125 : x.

V. If 125 men do a certain work when employed 12 hours a day, how many will do it when employed $7\frac{1}{2}$ hours a day?

The proportion is inverse, and the statement is 7·5 : 12 :: 125 : x.

Compounding these several ratios—

$$\left.\begin{array}{r@{\;:\;}r} 100 & 500 \\ 20 & 8 \\ 4 & 6 \\ 3 & 4 \\ 7{\cdot}5 & 12 \end{array}\right\} :: 125 : x.$$

$$\overline{100 \times 20 \times 4 \times 3 \times 7{\cdot}5 : 500 \times 8 \times 6 \times 4 \times 12 :: 125 : x.}$$

Or by cancelling—

$$x = \frac{5 \times 8 \times 6 \times 12 \times 125}{20 \times 7{\cdot}5 \times 3} = \frac{125 \times 32}{5} = 800 \text{ men.}$$

By taking all the dimensions of each trench at one statement the sum would have assumed this form,—

$$\begin{array}{lr@{\;:\;}rl} & 100 \times 20 \times 1{\cdot}3 & 500 \times 8 \times 2 & \multirow{2}{*}{\Big\} :: 125 : x.} \\ & \underline{3 \times 7{\cdot}5} & \underline{12 \times 4} & \\ \text{Cancelling} & 5 & 32 & :: 125 : x. \end{array}$$

Exercise CV.

1. If it takes 1,200 yards of cloth $\frac{5}{4}$ wide to make clothes for 500 soldiers, how much cloth $\frac{7}{8}$ wide will be needed to clothe 960 men?

2. A man travelling 15 hours a day, reaches a distance of 375 leagues in 20 days; how many hours per day must he travel to pass over 400 leagues in 18 days?

3. It takes 15 labourers, working 10 hours a day, 18 days to get through 450 yards of a certain work; how many men, working 2 hours a day, would, in 8 days, finish 480 yards of the same work?

4. If 7 horses be kept 21 days for £14, how many may be kept 10 days for £20?

5. If 850 men consume 240 quarters of wheat in six months, in what time would 3,230 men consume 1,820 quarters?

6. How long will it take 17 men to earn £50, if 12 men, in 6½ days, can earn 13 guineas?

7. If 20 sportsmen, firing 5 shots in 6 minutes, kill 500 birds in an hour, how many birds would be killed in an hour and a half by 10 sportsmen, firing at the rate of 3 shots in 5 minutes?

8. If 300 men could do a piece of work in 2·4 days, how many would do one-third of the same work in 12 days?

9. If the work done by a man, a woman, and a boy respectively, be proportioned as 3, 2, and 1; and if 9 men, 15 women, and 18 boys finish a certain work in 208 days, in what time would 15 men, 12 women, and 9 boys finish the same?

10. 250 men are set to work on a railway embankment, a mile and a half long, which they are expected to finish in 4 weeks. But at the end of one week it is found that they have only finished 520 yards; how many more men must be engaged to finish it in the required time?

11. If a mass of silver be worth £720,000 when silver is worth £3 17s. 6d. per pound troy, how much would it be worth when the current price is at the rate of 13·75 shillings for 2·5 ounces?

12. A party of 7 gentlemen, on a journey together, spend £150 in 3 weeks 4 days; what would be the expense, at the same rate, of another party, consisting of 11 persons, travelling for a fortnight?

13. If a plot of building land, 272 feet 6 inches long by 35 feet 8 inches broad, be sold for £76 10s. 9d., how much will a plot 283 yards 2 feet long by 74 feet 9 inches wide cost?

14. If 40 men can reap 400·6 acres in 12·75 days, how many acres ought 30 men to reap in 3·4 days?

SECTION V.—INTEREST, DISCOUNT, ETC.

369. INTEREST is money paid for the use of money.

The money lent is called the PRINCIPAL.

The interest allowed per annum on every £100 of the Principal is called the RATE PER CENT.

The sum of the Principal and Interest is called the AMOUNT.

370. Interest is a quantity which *varies directly* as the principal.

It also varies directly as the time for which it is lent.

Whenever principal and interest are alone referred to in a sum, the question is solved by Simple Proportion.

But when principal and time are both considered in relation to the interest, the problem is one in Compound Proportion.

371. As in nearly all problems under this Rule, the sums of money mentioned in the question can be expressed decimally, the answer, if obtained by the decimal method as far as the third place (301), will seldom be so much as one farthing wrong. It will be found very advantageous to use decimals throughout this Rule.

SIMPLE INTEREST.

372. In Simple Interest we have Principal, Rate per cent., Interest, and Time, to consider, and any three of these being given, we are required to find the fourth.

The following are examples of each form of question:—

373. *Example I.*—What is the interest on £272 at 3 per cent.?

Here the answer is to be interest.

∴ £3, which is the known interest on £100, is the third term.

And $3 : x$ is the ratio to be solved.

And by Rule of Three $100 : 272 :: 3 : x$.

$$\therefore x = \frac{272 \times 3}{100} = £8{\cdot}16 = £8\ 3\text{s}.\ 2\tfrac{1}{2}\text{d}.$$

This sum would ordinarily be worked thus:—

$$\begin{array}{rr} & 272 \\ & \underline{3} \\ \pounds & 8,16 \\ & \underline{20} \\ \text{s.} & 3,20 \\ & \underline{12} \\ \text{d.} & 2,40 \\ & \underline{4} \\ \text{f.} & 1,60 \end{array}$$

Answer, £8 3s. 2¼d.

For in every sum in Interest the statement takes the same form: £100 is always the first term, the principal is always the second term, and the rate of interest always the third. Division by 100 is most readily effected *by cutting off two figures from the right hand.*

Observation. — The answer obtained by the former method, £8 3s. 2½d., is evidently rather nearer the truth than this answer, and is much more conveniently worked.

374. *Example II.*—From what principal did £273 15s. arise in a year, at 4½ per cent. ?

Here the answer is to be principal. But cent., or £100, is the principal named in the sum, therefore $100 : x$ is the ratio to be solved.

Because the proportion between interest and principal is direct, ∴ the answer will be greater than £100;

And the statement is £4 10s. : $273{\cdot}75 :: 100 : x$.

$$\therefore x = \frac{273{\cdot}75 \times 100}{4{\cdot}5} = \pounds 6083{\cdot}\dot{3} = \pounds 6{,}083 \text{ 6s. 8d.}$$

375. *Example III.*—At what rate per cent. will £720 amount to £774 in a year?

Here the answer required is interest on £100.

But because £720 amounts to £774, the difference between these two, or £774 − £720, or £54, is the interest on £720.

∴ the statement is $720 : 100 :: 54 : x$.

$$\text{And } x = \frac{54 \times 100}{720} = \pounds 7{\cdot}5 = \pounds 7 \text{ 10s. per cent.}$$

376. Whenever the condition of time is introduced, the question is one in Compound Proportion.

Example IV.—What is the interest on £572 10s., at 4 per cent., for 6 weeks?

Here interest is to be the answer. ∴ $4 : x$ is the ratio required.

Two circumstances affect this ratio, principal and time.

I. If £4 is the interest on £100, what is the interest on £572 10s.?

The statement is 100 : £572 10s. $:: 4 : x$.

II. If £4 be the interest for 52 weeks, what is the interest for 6 weeks?

The statement is $52 : 6 :: 4 : x$.

Combining these two statements—

$$\left.\begin{array}{r} 100 : 572{\cdot}5 \\ 52 : \quad 6 \end{array}\right\} :: 4 : x.$$

$$\therefore x = \frac{572{\cdot}5 \times 6 \times 4}{100 \times 52} = \pounds 2{\cdot}642 = \pounds 2 \text{ 12s. } 10\tfrac{1}{4}\text{d}$$

377. *Observation.*—This sum might be done by finding the year's interest according to the last rule, and then establishing a simple proportion, thus:—

As one year *is to* the given time, *so is* a year's interest *to the* interest for the given time.

378. *Example V.*—What is the principal from which £270 arises as interest in $4\frac{1}{2}$ years, at $6\frac{1}{4}$ per cent.?

Here the answer is to be principal, and $100 : x$ is the ratio to be solved.

The two questions to be considered are,—

I. If £100 is the principal from which £6·25 arises, what is the principal from which £270 will arise? This is a case of Direct Proportion, and the answer required will be greater.

∴ the statement is $6{\cdot}25 : 270 :: 100 : x$.

II. If £100 is the principal from which a certain interest arises in one year, what is the principal from which the same interest would arise in $4\frac{1}{2}$ years? This is a case of Inverse Proportion, and the answer required is to be less than £100.

∴ the statement is $4{\cdot}5 : 1 :: 100 : x$.

Combining these two statements—

$$\left.\begin{matrix} 6{\cdot}25 & : & 270 \\ 4{\cdot}5 & : & 1 \end{matrix}\right\} :: 100 : x.$$

$$\text{And } x = \frac{270 \times 100}{6{\cdot}25 \times 4{\cdot}5} = £960.$$

379. *Example VI.*—At what rate per cent. would £1720 amount to £1978 in 5 years?

Here the answer required is interest, and $£1971 - 1720 = 258$ is the interest on the given sum, and $258 : x$ is the ratio to be determined.

I. If £258 be the interest on £1720, what is the interest on £100?
The proportion is direct, and the answer required is less.

∴ the statement is $1{,}720 : 100 :: 258 : x$.

II. If £258 be the interest for 5 years, what is the interest for 1?
This proportion is also direct, and the required answer is to be less.

∴ the statement is $5 : 1 :: 258 : x$.

Combining these two statements—

$$\left.\begin{matrix} 1720 & : & 100 \\ 5 & : & 1 \end{matrix}\right\} :: 258 : x.$$

$$\text{And } x = \frac{100 \times 258}{1720 \times 5} = £3 \text{ per cent.}$$

RULE TO FIND SIMPLE INTEREST.

380. Multiply the Principal by the Rate per cent., and divide by 100.

The other cases may be understood by the help of the following formulæ.

There are four items, viz., Principal, Interest, Rate per cent., and Time, any three of which being given, the fourth may be found.

Let P. = Principal, I. = Interest, R. = Rate per cent., and T. = Time.

I. When Principal, Rate, and Time are given, to find the Interest—

$$\text{I.} = \frac{\text{P. R. T.}}{100}$$

II. When Interest, Rate, and Time are given, to find the Principal—

$$\text{P.} = \frac{100 \times \text{I.}}{\text{R. T.}}$$

III. When Principal, Interest, and Time are given, to find the Rate—

$$\text{R.} = \frac{100 \times \text{I.}}{\text{P. T.}}$$

IV. When Principal, Interest, and Rate are given, to find the Time—

$$\text{T.} = \frac{100 \times \text{I.}}{\text{P. R.}}$$

Exercise CVI.

1. Find the interest on £357 12s. at 5 per cent. for 1 year.
2. Find the interest on £4098 at 4½ per cent. for 1½ year.
3. Find the interest on £729 16s. at 3¼ per cent. for 2 years.
4. Find the interest on £874 13s. at 2½ per cent. for 4 years.
5. Find the interest on £3096 10s. at 3 per cent. for 4¼ years.
6. Find the interest on £895 at 2¼ per cent. for 7 yrs. 2 months.
7. Find the interest on £3728 at 1½ per cent. for 3 years 7 weeks.
8. Find the interest on £8547 at 2·75 per cent for 8 years.
9. Find the interest on £3276 at 6⅛ per cent. for 5 years.
10. Find interest on £8097 at 7⅛ per cent. for 7 weeks.
11. Find interest on £1813 19s. at 3¼ per cent. for 9 months.
12. Find interest on £5208 16s. at 7½ per cent. for 11 wks. 3 days.
13. Find interest on £7109 18s. at 3¼ per cent. for 2 yrs. 3 months.
14. Find interest on £8297 13s. 6d. at 5½ per cent for 17 months.
15. Find interest on £8634 15s. at 7¼ per cent. for 173 days.

16. Find interest on £718 10s. at 3⅙ per cent. for 2 years 94 days.
17. Find interest on £8274 12s. at 6¼ per cent. for 7 years 8 weeks.
18. Find interest on £3274 at 3⅛ per cent. for 21 years.
19. Find interest on £8067 15s. 3d. at 7⅛ per. cent. for 6 yrs. 7 wks.
20. In what time will £723 15s. amount to £1280 at 3 per. cent.
21. In what time will £1072 16s. amount to £2000 at 5¼ per cent.?
22. In what time will £863 10s. amount to £1073 at 7¼ per cent.?
23. In what time will £427 amount to £500 at 2¼ per cent.?
24. At what rate per cent. will £832 amount to £998 8s. in 5 yrs.?
25. At what rate per cent. will £79 amount to £100 in 7½ years?
26. At what rate per cent. will £1016 amount to £1250 in 5 years 5 months?

27. At what rate per cent. will £729 amount to £850 in 6¼ years?

28. From what principal will £86 accrue as interest in 2 years at 5 per cent.?

29. From what principal will £274 accrue as interest in 4 months at 3 per cent.?

30. From what principal will £18 accrue as interest in 7 weeks at 6 per cent.?

31. From what principal will £259 10s. accrue as interest in 3½ years at 4¼ per cent.?

32. How long will it be before £375 put out at interest at 4 per cent. will realize a profit of £120?

33. What sum must I invest at 3¾ per cent. so as to secure an annual income of £250?

34. If after lending £1560 for 3 years I receive the sum of £1747 4s. in repayment, at what rate per cent has the money been invested

35. What time must elapse between the time of placing £28 in a Savings Bank and of taking out £40 19s., supposing interest is at 2½ per cent.?

36. If I invest £1673 10s. at £3 15s. 6d. per cent. for 18 years 3 months, to what sum will it amount?

37. In what time will a sum of money double itself at 3½ per cent. per annum simple interest?

38. To what sum will an investment of £1,726 10s. amount in 15¾ years at 2½ per cent.?

39. Find the simple interest on £926 15s., for 200 days at 7 per cent. per annum.

COMPOUND INTEREST.

381. When the interest of money is added to the principal at certain periods, and afterwards interest is calculated on this amount, the money is said to be put to Compound Interest.

Every sum in Compound Interest consists of a series of sums in Simple Interest, a separate calculation being required for each of the periods mentioned in the sum.

Example.—What is the compound interest on £628 at 5 per cent. for three years?

By Rule (373) $\frac{628 \times 5}{100}$ = £31·4=£31 8s. =interest for the 1st year.

£628 + £31·4=£659 8s. =principal at the beginning of the 2nd year.

$\frac{659{\cdot}4 \times 5}{100}$ = £32·97 = £32 19s. 5d. = interest for the 2nd year.

£659·4 + £32.97 = £692·37 = £692 7s. 5d. = principal at beginning of 3rd year.

$\frac{692{\cdot}37 \times 5}{100}$ = £34·618 = £34 12s. 4¼d. = interest for 3rd year.

£692·37 + 34·618 = 726·988 = £726 19s. 9¼d. = amount at end of 3rd year.

Hence £628 has become £726 19s. 9¼d. in the three years, and the total interest which has accumulated on it is—

£726 19s. 9¼d. − £628, or £98 19s. 9¼d.

382. A somewhat readier method of obtaining the result is that of finding the compound interest on £100 for the given time, and then solving the original question by a Proportion sum.

Thus £5 = interest for the first year.

∴ £105 = principal at beginning of 2nd year.

$\frac{105 \times 5}{100}$ = £5·25 = £5 5s. = interest for 2nd year.

∴ 105 + 5·25 = £110·25 = £110 5s. = principal at beginning of 3rd year.

But $\frac{110{\cdot}25 \times 5}{100}$ = £5·512 =£5 10s. 3d. = interest for 3rd year

And £110·25 + £5·512=£115·762 = £115 15s. 3d. = principal at end of 3rd year.

Therefore £100 has accumulated to £115 15s. 3d. in three years, and £15 15s. 3d. is the compound interest on £100 for the given time.

We have now the following proportion :—

If £15 15s. 3d. be the compound interest on £100, what will be the compound interest on £628 for the same time and rate?

The statement is 100 : 628 ∷ £15 15s. 3d. : x.

$$\text{And } x = \frac{15\cdot762 \times 628}{100} = £98\cdot988 = £98 \text{ 19s. } 9\tfrac{1}{4}\text{d.}$$

RULE FOR COMPOUND INTEREST.*

383. Find the simple interest for each of the periods mentioned in the sum, and add their results successively to the principal. Or,

Find the compound interest on £100 for the given time, and then solve the question by Proportion, as in the example.

EXERCISE CVII.

1. Find compound interest on £500 at 4 per cent. for three years.
2. Find compound interest on £720 for three years at 4½ per cent.
3. Find compound interest on £1150 for 2½ years at 5 per cent.
4. Find compound interest on £484 for 5 years at 3½ per cent.
5. Find compound interest on £1250 for 2 years at 6 per cent.
6. Find compound interest on £7084 for three years at 3¼ per cent.
7. Find compound interest on £2257 for 2½ years at 2¼ per cent.
8. Find compound interest on £1097 for 5 years at 3½ per cent.
9. Find compound interest on £2384 for 4 years at 4 per cent.
10. Find compound interest on £5063 for 3 years at 2¼ per cent.
11. To what sum will £187 amount in 5 years at 3 per cent. compound interest?
12. What is the compound interest on £1200 at 6 per cent. for 2½ years, the interest being paid half-yearly?
13. To what sum will £150 amount in 3¼ years at 4 per cent., the interest being paid quarterly?
14. If on the birth of a child £100 were invested at 5 per cent. compound interest, payable yearly, to what will it amount on his coming of age?

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* The more difficult problems in compound interest are generally worked by the help of logarithms. See 553.

DISCOUNT.

384. *Discount* is a deduction made from a debt which is paid before it is due.

Questions of this sort occur in this Rule :—

A debtor is bound to pay £450 at the expiration of a year and a half from this date ; what ought he to pay *now* to clear himself if interest is 4½ per cent. ?

Observe here that we must not calculate interest on the £450, for that is not the principal, but the *amount* to which his present debt will have accumulated if he leaves it unpaid for a year and a half. Whatever he owes at this moment is the principal, and this with the interest upon it at 4½ per cent., will amount to £450 in a year and a half. We have therefore to separate £450 into two parts, so that one shall be the interest upon the other at 4½ per cent. for 1½ years.

As in the second case of compound interest, it will be convenient to take £100 as a standard, and find what it would amount to at the same rate and time.

Now $\frac{100 \times 4{\cdot}5 \times 1{\cdot}5}{100} = £6\ 15\text{s.}$ = interest on £100 for 1½ years.

∴ £100 present debt would become £106 15s. a year and a half hence at 4½ per cent.

We have now the following proportion :—

If £100 will become £106 15s. in a certain time and at a certain rate, what sum will amount to £450 at the same rate ?

The statement is $106{\cdot}75 : 450 :: 100 : x$.

$$\text{And } x = \frac{450 \times 100}{106{\cdot}75} = £421{\cdot}545 = £421\ 10\text{s.}\ 11\text{d.}$$

385. This answer gives the Present Value, and the difference between this and £450 is called the Discount. Or the discount itself might be found by the following proportion :—

If £6 15s. is the discount which should be deducted from £106 15s. what is the discount to be deducted from £450 ?

The statement is £106 15s. : 450 :: £6 15s. : x.

$$\text{And } x = \frac{450 \times 6{\cdot}75}{106{\cdot}75} = £28{\cdot}455 = £28\ 9\text{s.}\ 1\tfrac{1}{4}\text{d.}$$

386. The difference between the calculation of Interest and Discount is this, that in the one the principal is generally given, and we are required to find the interest; but in the other a sum of money is given which includes both principal and interest, and we are required to separate it into those wo parts. This latter problem is always to be solved by choosing a sum of money, finding what it would amount to in exactly the same circumstances as the unknown principal, and then establishing a proportion. For convenience we generally select £100 for this purpose, but any other sum of money might be the basis of the proportion.

387. *Example.*—What sum of money is that which, after lying in the Bank 7 years, when money is at 3 per cent., will amount to £5630?

We will choose £50 as the basis of the proportion:—

$$\frac{50 \times 7 \times 3}{100} = £10{\cdot}5 = £10 \text{ 10s.} = \text{the interest on £50 for the given time.}$$

$\therefore$ £50 would become £60 10s. in the same circumstances.

If a principal of £50 would become £60 10s. in 7 years at 3 per cent., what principal is that which will become £5630 at the same rate and time?

The answer required is principal, and 50 : x is the ratio to be solved.

$$£60 \text{ 10s.} : 5630 :: 50 : x.$$

$$\text{And } x = \frac{5630 \times 50}{60{\cdot}5} = £4652{\cdot}892 = £4652 \text{ 17s. } 10\tfrac{1}{4}\text{d.}$$

This answer is necessarily the same as if 100 had been taken, for then the statement would have been—

$$£121 : 5630 :: 100 : x.$$

$$\text{And } x = \frac{5630 \times 100}{121} = £4652{\cdot}892 = £4652 \text{ 17s. } 10\tfrac{1}{4}\text{d.}$$

388. BILLS AND PROMISSORY NOTES are written engagements on the part of a debtor to pay a certain amount on some future day. A *Bill* is drawn by the person to whom the money is due, and accepted or signed by the debtor. By accepting the bill he becomes legally liable for the amount when the time expires. A *Promissory Note* is simply signed by the party promising to pay. Three days are allowed by law beyond the date specified in the document; thus a Bill or

Promissory Note made payable on the 3rd of September is not legally presentable until the 6th, unless this day should be Sunday, in which case it must be presented on the preceding day.*

Suppose A holds a bill by which B is bound to pay £500 this day three months, and A wishes to realize his money immediately. It is clear that if he parts with his claim or sells the bill, he ought to receive in lieu of it such a sum of money as, if put out at interest for three months, would amount to £500 by the end of that time. Many persons, as bankers and bill-discounters, are ready to negotiate in such matters and to buy bills before they are due, provided that the credit of the acceptor is good, and he is considered likely to pay when the bill falls due. But, in fact, the true discount is never calculated on bills, but the *interest* on the sum for which the bill is drawn is deducted from the whole sum, and is called the discount. The discounter of the bill evidently derives a small advantage from this arrangement. For if 4 per cent. is the rate of interest, and he pays A £495 for the bill, deducting £5, or three months' interest on £500 at 4 per cent., the discounter really receives rather more than 4 per cent. for the money which he advances. He obtains interest on the sum of £500, when, in fact, he only lends £495; and the sum which he lends, if put out at interest, would *not* amount to £500 in three months.

Tradesmen often make an allowance to such of their customers as choose to pay ready money. This allowance is called Discount, but is really calculated as Interest, the per-centage being found on the whole sum at a certain rate, and deducted from it.

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* The following is a form of Bill:—

To Mr. C. D., Liverpool. Gresham-street, London, Sept. 8, 1878.

Six months after date pay to me or my order the sum of one hundred and fifty pounds, value received. A B.

£150 0s. 0d.

C. D. *accepts* the bill by writing his name either under that of A. B., the *drawer*, or across the bill.

A Promissory Note has the same force, and subjects the debtor to the same obligation. Its form is generally as follows:—

Liverpool, Sept. 8, 1878.

Six months after date I promise to pay to Mr. A. B. or his order the sum of one hundred and fifty pounds. C.D.

No other signature is required here than that of C. D., the debtor, but such a note is usually endorsed by the creditor.

RULE FOR DISCOUNT AND PRESENT VALUE.

389. Take £100, find what it would amount to at the same rate and time, and state the proportion as follows :—

As £100 + *its interest for the given time* is to *the given amount*, so is £100 to the present value.

General Formula.—If A = amount, and I = interest on £100—

$$\text{Then } \frac{100\,A}{100 + I} = \text{Present Value.}$$

Or if the discount be required—

As £100 + *its interest* is to *the given amount*, so is *the interest on* £100 to the answer.

$$\textit{General Formula.}\text{—} \frac{A \times I}{100 + I} = \text{Discount.}$$

In the following Exercise the expression *true discount* refers to the calculation made in this way, while *ordinary discount* means the deduction actually made in business.

Exercise CVIII.

1. What sum will discharge a debt of £720, due a year and a half hence, at 4 per cent.?

2. What sum put out at interest will amount to £1,310 14s. in six years at 4⅔ per cent. ?

3. If a legacy of £1,200, less 5 per cent. duty, is to be paid to a person whose age is 17, when he becomes 24 years old, what sum paid to him *now* would be equivalent to it, interest of money being at 5 per cent.?

4. What is the difference between the ordinary bankers' discount on a bill of £470 due six months hence, at 5 per cent., and the true discount ?

5. A tradesman accustomed to give credit is offered ready money for an account of £58 14s. ; what is to be paid if the discount is 5 per cent. ?

6. Find the true discount on £107 5s., payable at the end of six months, at 3½ per cent. ?

7. What is the exact present value of a debt of £572, due eight months hence, at £3·75 per cent. simple interest?

8. How much money must I invest at 7 per cent. in order that at the end of four and a half years I may be worth £5,000?

9. Find the difference between the true and the ordinary discounts in the case of the following bills—

£450 drawn on March 1, payable on June 1, at 4 per cent.
£1000 drawn on June 18, payable on August 18, at 5 per cent.
£1728 drawn on Sept. 27, payable on Dec. 31, at 3½ per cent.

10. What is the difference between the true and the ordinary discount on the following bills?—

£2,347 drawn on Aug. 26, payable on Jan. 15, at 2¼ per cent.
£6274 drawn on Feb. 13, payable on June 16, at 3¾ per cent.
£240 drawn on April 3, payable on July 1, at 2¼ per cent.

11. By how much does banker's discount exceed the true discount in the case of the following bills?—

£27 drawn on Jan. 6, payable on April 28, at 3¼ per cent.
£48 drawn on Dec. 5, payable on Jan. 8, at 5 per cent.
£623 drawn on Dec. 28, payable on April 3, at 4 per cent.

12. If a bill for £100 drawn on the 24th of June, and payable on the 13th of January, be discounted on the 27th of September, what will be the true discount at 4 per cent.?

13. If a certain investment amounts to £7680 in 3 years, at 2½ per cent., what is the sum invested?

14. How much must be deposited in the 3 per cents. in order to amount to £5675 in 4 years?

15. A tradesman sells goods, either for cash at a discount of 2½ per cent., or at 4 months' credit. If I buy goods worth £31 10s., and pay half of the sum in ready money, taking credit for the rest, what will the goods cost me?

16. What sum invested at 3½ per cent. will amount to £1,000 in 5½ years?

17. At what rate per cent. compound interest will 70 guineas amount to £114 16s. 10½d., in two years?

18. What sum of money invested at 7 per cent. per annum will amount to £962 2s. 8½d., in 200 days?

STOCKS AND SHARES.

390. *Public Funds.*—In the reign of William III. the Government of England laid the foundation of our present National Debt by borrowing money from private persons, which was employed for the necessities of the State.* In order to induce capitalists to dispose of their money in this way, interest was offered at a certain fixed rate per annum. At different periods since that time foreign wars and other national emergencies have made it necessary for the Government to borrow more money of individuals; and the total sum which is thus due by the State exceeds seven hundred millions of pounds sterling.†

Price of Stock.—Every person whose money has been thus invested receives interest regularly every half-year, at a certain rate per cent., and also holds an acknowledgment from the Government of the debt. These acknowledgments may be transferred from one name to another, and may be bought by any person who chooses. Like everything else which is marketable, they fluctuate in value, becoming dear when there is a general disposition on the part of monied people to buy them, and becoming cheap when many of the holders are anxious to sell.

When the prospects of the country are unusually good, and there is a greater probability than usual that the debt will be reduced; or when money is so abundant that persons possessing it cannot readily find other investments, and so are glad to get the 3 or 3½ per cent. which Government allows, buyers of stock are numerous, and prices advance. But the prospects of war, the chances of a bad harvest, or such a scarcity of money as makes people anxious to employ their capital in other ways, will lessen the number of buyers, and cause the stock to become cheap. The extent of speculation, and many other circumstances, also affect the value of the funds, so that it is seldom that prices remain perfectly stationary even for two consecutive days. The highest price ever attained by £100 stock was in 1752, when it was quoted at £106 7s. 6d.; and the funds reached the lowest point ever known in September, 1797, when £100 could be bought for £47 12s. 6d.

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* "See Macaulay's "History of England," vol. iv., p. 319. Up to this period money had often been borrowed by our sovereigns, but generally on the security of some special tax. A government in want of money would mortgage the "tonnage and poundage," for example, and if it failed to produce the necessary sum, the lenders lost money by the transaction and were never repaid. But it was in this reign that the national credit was for the first time pledged to the lenders, and that the *permanent* debt was established.

† At the accession of George I. the debt amounted to £54,000,000. Sir Robert Walpole's sinking fund reduced it by 1739 to £46,954,000. But in 1763, after the Seven Years' War, the debt amounted to £146,000,000, and at the close of the American War to £257,000,000. Mr. Pitt's sinking fund of 1786 again slightly reduced the debt during the peace by nearly five millions; but the expenses of the subsequent war with Napoleon enormously increased it, so that at the end of 1815 the year of the battle of Waterloo the public debt had reached £836,255,000. By 1875 it had been reduced to £775,348,686. In 1885 terminable annuities to the extent of about fifty millions will expire; and the annual charge, which in 1875 amounted to £27,000,000, will be reduced by about five millions and a half. Nearly the whole of the funded debt carries interest at 3 per cent.

When the market price of £100 stock is £100, the funds are said to be at *par*. They are usually below par; and in ordinary times prices fluctuate between 91 and 98.

The person in whose name the stock stands at the end of each half-year receives the dividends, and therefore a purchaser has the benefit of the interest on the stock he buys, from the last day of payment to the day of transfer. Hence, all other things being equal, the approach of the dividend day causes the price of stock to advance, and after the dividend has been paid prices experience a slight decline.

391. *Rate of Interest.*—When a person takes a share in the National Debt he is said to purchase stock. If he does this when funds are low, he is credited with a larger sum than he actually invests; and as he receives £3 or £3 5s. per cent. on this nominal value, the interest actually accruing on his capital is greater than the nominal interest. The price of the funds is therefore a good measure of the rate of interest, and of the general abundance of money,—for interest is high when funds are low, and *vice versa*. If funds rise in price, and a man sells his shares at a higher value than that at which he bought them, he may obtain profit by the transaction just as in ordinary trade. Many persons make a business of the purchase and sale of stock with a view to profit in this way, by speculating on the probabilities of a rise or fall in the funds.*

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* For example, A engages to sell B £1000 stock for £900 on a certain fixed day, perhaps three months hence. A possesses, in fact, no such stock, but he has reason to believe that by the time mentioned the price of stock will be below 90, and in that case he will gain a profit,—for should funds be at 88, he will be able to buy £1000 stock for £880, and so to gain £20 by the transaction. B, however, engages to buy in the hope that

392. *Foreign Loans.*—Sometimes foreign governments desire to raise a loan, and invite all the capitalists of Europe to take shares in it. Bonds, or engagements to pay interest until the money is returned, are issued and sold in the money markets. If the credit of the government wishing to borrow be generally considered good, and the interest offered be high, many persons will become eager to take up portions of the loan; the prices of the stock advance, and the actual interest on the investment is proportionately diminished. The market price of foreign bonds or securities is of course affected by the same circumstances as that of the English funds.

393. *Commercial Companies.*—Shares in railway, canal, or other public companies are also bought and sold in the share market, and are regulated by the same laws. For example, suppose a company is started requiring a capital of £1,000,000, and this sum is distributed in the form of 100,000 shares of £10 each. If the prospects of the company are generally considered very good, and it is likely to realize a handsome profit on its capital, many persons will desire to become shareholders, and the shares will rise in value. Perhaps a £10 share will be sold for £12: in this case the value of the company's shares is said to be *£2 premium:* but if, on the other hand, persons are unwilling to join in the undertaking, the shares will probably be sold at a *discount*, *i.e.*, at less than their nominal value.

394. *Transfers of Stock.*—Transfers of invested capital from one form of stock to another are very common. Whenever this is done, the nominal value of the given sum is in inverse proportion to the market price of the stock. Thus: if I have a sum of money invested in Russian securities, which are at 57, and I transfer it to the English funds at 88½, the sum standing in my name in the latter will be as many times less than in the former as 57 is less than 88½. For if I were to sell the Russian stock, I should only realize £57 for every nominal £100, and with this sum I could not purchase nearly so large an amount of stock at 88½. On the other hand, if I sell out stock at a premium, and buy stock at a discount, the nominal amount of stock I hold will be greater.

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funds will be at a higher price than 90 and expects to gain a profit by purchasing stock at that price and selling it at the market rate. In the slang of the Stock Exchange, A would be called a *Bear*, and B, who speculates on the chance of a rise, is called a *Bull*. Such bargains, though very common, are a sort of gambling, and are not recognized by law.

395. *Brokerage.*—The persons who effect the necessary sales, and whose business it is to transfer stock and shares from one name to another, are called *Brokers*, and the sum they receive for their trouble is called *Brokerage*. The commission thus paid to Stockbrokers is always ⅛ per cent., or 2s. 6d. on every £100 stock in the funds, but is higher for other stock. It must always be added to the cost of purchasing stock, and subtracted from the receipts of one who sells.

The principles of proportion apply to all calculations of this kind. The following are examples of the sort of questions which occur in these rules :—

396. *Example I.* How much stock in the 3 per cents. can be purchased for £1240 when the price is 89¾ ?

Here is a case of Inverse Proportion, for the lower the market value of the stock the more can be purchased for a given sum.

The answer required is to be as many times greater than £1240 as 89¾ is less than 100.

$\therefore$ the statement is $89\frac{3}{4} : 100 :: 1240 : x$.

$$\text{And } x = \frac{1240 \times 100}{89{\cdot}75} = 1381{\cdot}615 = \pounds 1381 \text{ 12s. } 3\tfrac{3}{4}\text{d.}$$

397. *Example II.*—What will it cost to purchase £1050 stock when funds are at 93¼ ; brokerage at ⅛ per cent. ?

Here the expense of brokerage must be added to the value of the stock, as it will have to be paid by the purchaser.

The case is one of Direct Proportion, for the lower the value of £100 the less money will be required to purchase any given nominal amount.

$\therefore$ the statement is $100 : 93{\cdot}375 :: 1050 : x$.

$$\text{And } x = \frac{1050 \times 93{\cdot}375}{100} = 980{\cdot}4375 = \pounds 980 \text{ 8s. 9d.}$$

398. *Example III.*—What sum will stand in my name in the 3 per cents. at 81¼ if I transfer £1450 from Russian stock at 54½ ?

Here is a case of Inverse Proportion, because the lower the price of the stock the greater the nominal value of a given sum of money.

$\therefore$ the statement is $81{\cdot}25 : 54{\cdot}5 :: 1450 : x$.

$$\therefore x = \frac{1450 \times 54{\cdot}5}{81{\cdot}25} = 972{\cdot}615 = \pounds 972 \text{ 12s. } 3\tfrac{3}{4}\text{d.}$$

EXERCISE CIX.

1. What sum invested in the 3 per cents. when they are at 98¼ will have the nominal value of £7,268?

2. What annual income should I derive from an investment of £3,500 in the 3¼ per cents. if I buy the stock at 91½?

3. What is approximately the price of stock when I can buy £3,158 for £2,754 10s.?

4. If I lay out £840 in the purchase of stock at 79½, and sell out at 85⅝, what do I gain?

5. If I purchase 120 Bank Annuities at 90⅜, at what price must I sell them so as to gain £150?

6. What is the difference between the annual income arising from the investment of £2,150 in the 3¼ per cents. at 87⅛, and that from the same sum in the 3 per cents. at 86½?

7. A person invests £1,248 in the 3 per cents. at 95⅛; what will be his net half-yearly dividend after deducting 7d. in the pound per annum for income tax?

8. What would be realized by selling £8,296 10s. stock at 88⅝?

9. If funds are at 82⅛, what must be given for £1,250 stock?

10. How much should I gain by purchasing £700 stock at 97⅝, and selling out at 101¾?

11. When the 3 per cents. are at 89¼, at what rate may the same quantity of stock be purchased in the 3½ per cents. with equal advantage?

12. A person invested £3,000 in the 3 per cents. when they were at 76⅛; what was his annual income?

13. How much stock at 91⅛ can be bought for £768, ⅛ per cent. being charged for commission?

14. Suppose I lay out £1,270 in the 3 per cents. at 92½, and sell out after allowing the interest to accumulate for two years, and find myself the richer by £147 10s.; at what price do I sell out?

15. What ought to be the current rate of interest per cent. when the 3¼ per cent. stock is at 92⅛?

16. If the current interest of money be 4 per cent., at what price ought the 3½ per cents. to be quoted?

17. What is the actual difference in interest between the 3 per cents. at 87⅝ stock, and the 3½ per cents. at 92¼?

18. If I invest £5,000 in the following manner; viz., three-fourths of it in the 3¼ per cents. at 92⅝, one-tenth of it in the threes at 89¼, and the rest in 4 per cent. stock at 98⅜, what annual income shall I derive from the whole investment?

MISCELLANEOUS APPLICATIONS OF THE TERM PER CENT.

399. ONE HUNDRED is employed as a standard of proportion in many other cases besides Interest; *e. g.*—

Profit and loss. Tradesmen measure their gains and losses on the whole of the capital they invest, by a certain sum *per cent.*, or on every £100 of their outlay.

400. *Commission.* Travellers, agents, collectors, and others, who are employed by a firm or a company, either to receive money or to extend business, are usually paid at the rate of so much *per cent.* on every £100 which passes through their hands.

401. *Insurance* against casualties, such as premature death, fire, or the loss of a ship at sea, is always calculated at a certain fixed rate per £100 on the sum insured. Thus: if a man at a certain age (say 40) desires to insure his life, a calculation is made founded on the average rate of mortality among persons of that age, and the insurer is charged with such an annual sum as, if he lives to the average age, will repay the insurance company and give them a reasonable profit. So, also, if a house or warehouse has to be insured against fire, the probability of the house taking fire is calculated from the ascertained number of fires in proportion to the total number of houses and warehouses; and this is determined at the rate of so much per £100.

Insurances of life as well as of houses, warehouses, &c., are usually effected by a company, as such dealings are necessarily on a large scale, and involve too great a risk for private persons to incur. But the insurance of vessels at sea is often undertaken by private persons who are called underwriters. It is thus effected. A is the owner of a ship whose cargo is worth £5,000; he desires to indemnify himself against the loss of the ship at sea; he therefore proposes to various underwriters to take shares in the insurance. One will perhaps take £500 worth of the risk, and another will engage himself to the extent of £1,000: each to receive a sum by way of insurance proportioned to the amount of risk he undertakes; and in the event of the loss of the ship, each will be bound to pay to the owners the sum for which he became responsible.

402. *Observation.*—It is to be noticed here that in the case of human life, of fire, of shipwreck, or of loss of life by railway, the amount of money paid as insurance is regulated by the *probability* of some

particular event. This must in every case be determined by past experience. Suppose, for example, it is proved, on referring to the history of past years, that one ship in 200 of all which go from London to the West Indies founders or is lost; then the underwriter is bound to assume that the chance of any particular ship reaching its destination safely is as 199 to 1. He therefore would be safe if he insured every ship going to the West Indies at $\frac{1}{200}$ part of the value of the cargo, because the doctrine of chances would justify him in expecting that 200 shipowners would pay him insurance for every *one* whom he would have to compensate for loss. By carefully keeping registers of casualties, the amount of risk incurred by life or property, either on the sea or in travelling, may be pretty accurately measured, and all insurance tables are founded on such registers. In all these the proportion is reckoned at so much per cent.

403. *Statistical tables* respecting population, employments, education, &c., are always constructed on the basis of 100, and are calculated at *per cent.* Thus: if it be found that in a country containing 2½ millions of inhabitants, a certain number, say 400,000, are employed in agriculture, it is usual to express this fact by stating that a certain number *per cent.*, or out of every 100 of the population, are so employed.

In every such case the number 100 is only assumed for convenience as the standard of proportion, and the principles of the Rule of Three are applicable.

404. *Example I.*—A man invests £2340 and makes £2587 of it; what does he gain per cent.?

Here his total gain is £2587 — 2340 = £247.

But the gain on £100 will be less than on £2340.

Therefore the statement is $2340 : 100 :: 247 : x$.

$$\text{And } x = \frac{247 \times 100}{2340} = 10\cdot\dot{5} = \text{£}10 \text{ 11s. } 1\tfrac{1}{4}\text{d.}$$

405. *Example II.*—An agent who sells an estate worth £12,346 receives a commission of 3s. 6d. per cent.; how much does he get by the transaction?

Here the statement is $100 : 12346 :: \cdot175 : x$.

$$\text{And } x = \frac{12346 \times \cdot175}{100}, \text{ the answer being in pounds.}$$

$$\text{Or } x = \frac{12346 \times 35}{100} = \text{ the answer being in shillings.}$$

406. *Example III.*—In a country containing a population of 14,500,000; 3,270,000 derive their subsistence from agriculture, 4,820,000 from trade, 2,748,000 from manufactures, 1,125,000 from professions, 975,000 are persons of independent means, and the rest are paupers; what is the per centage of population in each class?

Here the proportions in all cases are direct, and the statements are—

14500000 : 100 :: 3270000 : per centage engaged in agriculture.
14500000 : 100 :: 4820000 : per centage engaged in trade.
14500000 : 100 :: 2748000 : per centage engaged in manufactures.
14500000 : 100 :: 1125000 : per centage engaged in professions.
14500000 : 100 :: 975000 : per centage of independent persons.
14500000 : 100 :: 1562000 : per centage of paupers.

It will be found that all these answers added together will make 100, and will represent the proportions in which an average 100 of the population are distributed among the various classes of society.

Exercise CX.

1. In a school of 250 children, 44 per cent. are learning geography, 36 per cent. are learning grammar, 12 per cent. cannot read, and only 4 per cent. have advanced as far as algebra. What are the actual numbers of each?

2. An agent who is paid 2¾ per cent. on all the money he collects, receives £57 as commission; how much has he collected?

3. There are two schools, one containing 650, and the other 340 children: 5 per cent. of the former are generally absent, and 7·5 of the latter; what is the average attendance in each?

4. Air is composed of 3 gases, 75·55 per cent. being nitrogen, 23·32 oxygen, and 1·13 carbonic acid. In a chamber containing 3,274 cubic feet, how much is there of each gas?

5. In a certain year the total number of persons killed on all the railways of the United Kingdom was 216. Of these, 32 were passengers, 120 were railway servants, and the rest were trespassers. Find the nearest per centage of each class.

6. In the same year the total number of journeys made on the railways was 89,145,729. Express the per centage of passengers who lost their lives.

7. In 1831 the population of England and Wales was 13 896,797; in 1841 it was 15,914,148; in 1851 it was 17,927,609; in 1861, 20,066,224; and in 1871 22,782,266. What was the ratio of increase per cent. during each decennial interval?

8. In 1851 the population of the city of London was 129,181; in 1861 it was 113,387: what was the decrease per cent.?

9. Of 138,918 persons convicted of various offences during 5 years, 30·66 per cent. were unable to read and write; 58·89 per cent. could do both fairly; and the rest had received a decent education: what approximately were the numbers in each class?

10. The number of children belonging to all the public schools in England at the census of 1851 was 1,407,569; the number belonging to all the private schools was 700,904. Of the former, only 1,115,327 were actually present on the day of the census; of the latter 639,739 were in attendance: compare the per centage of attendance in public schools with that in private.

11. At the same time the population of England and Wales was 17,927,609; what per centage of the people was under tuition?

12. What is the insurance on £7,285 at £2 7s. 6d. per cent.?

13. If the rate of insurance be £1 6s. 6d. per cent., for how much is a person insured who pays an annual premium of £29 16s. 3d.?

14. If a warehouse contains goods worth £17,230, and is only insured for 86·3 per cent. of its value, what sum would be lost in case of its destruction by fire?

15. By selling 26 yards at 3s. 4½d. per yard, a draper gains 6s. 6d. What was the prime cost per yard, and what is the gain per cent.?

16. If a dishonest fruit vendor uses a weight of 14·76 oz. for 1 lb., and professes to sell his goods at the cost price, what does he gain per cent.?

17. If a grocer mixes 17 lbs. of tea worth 4s. with 25 lbs. worth 4s. 8d. and sells the whole at 5s. 4d. per lb., what is his total gain, and his profit per cent.?

18. By selling goods for £483 15s. a tradesman gets a profit of 7½ per cent.; what did he give for them?

19. If 1 cwt. 3 qrs. 4 lbs. are purchased for £50, what should be the retail price per lb. to give a profit of 5 per cent.?

20. Suppose a man insures his life for £1,000 at the yearly rate of £60, and dies immediately after he has paid his fifth annual premium; how much does the company lose by the transaction, reckoning money worth £5 per cent compound interest?

21. What amount of capital is that which, after being employed at ½ per cent. simple interest for 4 years, becomes £5,000?

Questions on Ratio and Proportion.

What is meant by *absolute* and *relative* magnitude? What is the usual standard of comparison? Distinguish Integral from Fractional, and both from Proportional Arithmetic. In how many ways may two magnitudes be compared? Which is the more usual of the two? Give a name to both. Define Ratio. Show how the definition given in the common translation of Euclid is open to objection. What sign is employed to indicate ratio? Define antecedent and consequent. Show how every sum in Multiplication and Division may be regarded as a sum in Proportion.

Why should the words "of like kinds" form part of the definition? If ratio be itself a magnitude, is it abstract or concrete? Why? How far do the truths concerning fractions apply to ratio? Give examples. What is the effect on a ratio of diminishing its consequent? of increasing it? of diminishing its antecedent? of increasing it? of increasing both? of diminishing both? Give numerical illustrations.

Define Proportion. Make three proportions, and explain how they illustrate your definition. How far do the words "of like kinds" apply in this case? Why? By what test can you determine whether four numbers are in proportion? Will the test apply in the case ot incommensurable magnitudes? Why? Prove in three ways that if $w : x :: y : z$, then $wz = xy$. What practical use is made of this truth in arithmetic?

How is any one term of a proportion to be determined when the other three are given? Suppose the product of any two numbers equals the product of any other two, what inference can you deduce from the fact? State the principle, prove it, and give two examples.

When is a ratio expressed in its lowest terms, and why? Give an example. How many other conclusions can you deduce from the fact that four numbers are in proporportion? Give a reason in each case. If I multiply every number in a proportion by itself how do I know that the resulting numbers are in proportion? Of what general truth is this a special example?

What is the purpose of the Rule of Three? Why is it so called? In the solution of a sum in Simple Proportion what two things have to be done? How are we to do the first? the second? What is Direct Proportion? Give other examples not mentioned in the text. Define Inverse Proportion. Give examples. How does a fraction illustrate the difference between Direct and Inverse Proportion? Take the two fractions $\frac{4}{11}$ and $\frac{4}{13}$, and show what principle explains their relation to each other.

How is a Rule of Three sum to be stated? Why? How is it to be solved after stating Why? When I multiply the second and third terms together and divide by the first, what truth is taken for granted?

When and how are ratios compounded? Give examples. What operation in Fractions resembles the composition of ratios? On what principle does the composition of ratios depend? Show how the principle applies. Into what parts may every problem in Compound Proportion be resolved? Take the first four sums in Compound Proportion, and resolve each of them into two distinct sums in Simple Proportion. Why should the two statements be combined?

Define Interest, Principal, Rate per Cent. Show how far Interest illustrates Direct Proportion, and in what cases Inverse Proportion occurs. Give in symbolical form. the rules for finding interest, principal, rate, or time. What is Compound Interest? Give an example, and the rule for working it. Why should we use the decimal method throughout this Rule? What is Discount, and how does it differ from interest? To what extent is this rule actually used in business?

What are Stocks? Explain why they fluctuate in value. Show how the price of Stocks is influenced by the current rate of interest. What is Brokerage? Give an example. Give some other examples of the use of the term *per cent.*

State the following sums: —

Find the interest on £a at b per cent. for c months, and also the discount.

What will a articles cost at the rate of b shillings for n articles?

If a men dig b cubic feet in c days, how many will do e cubic feet in m days?

If stocks are sold at a, how much can I buy for £b?

For what sum can I buy £a stock when the price is b?

If a cwt. of goods are carried b miles for a certain sum, to what distance would c cwt. be carried for the same sum?

Express in words the truths contained in the following formulæ:

If $a : b :: c : d$ Then $a ; c :: b : d$, and $c : a :: d : b$.

Also $a \pm b : b :: c \pm d : d$, and $a \pm c : b \pm d :: a : b$.

And $ma ; mb :: c : d$, and $a : b :: \frac{c}{n} \frac{d}{n}$

And $a^n : b^n :: c^n : d^n$, and $ad = cb$

MISCELLANEOUS EXERCISES ON PROPORTION.

1. If 7 men can reap 6 acres in 12 hours, how many men will reap 15 acres in 14 hours?

2. To what sum will £720 16s. amount in 3 years at 4 per cent. compound interest?

3. By how much does the compound interest on £527 for 4½ years at 2½ per cent. exceed the simple interest on the same sum for the same time?

4. If 800 soldiers consume 5 barrels of flour in 6 days, how many will consume 15 barrels in 2 days?

5. Find the discount of £125 8s. for 6 months at 3½ per cent.

6. What is the difference between the interest on £135 7s. 6d. for 9 months at 4 per cent., and the discount on the same sum?

7. What sum will be left out of the amount realized by selling £1000 of stock standing at £91$\frac{1}{10}$, if enough be set aside to clear £43 12s. in 1⅜ years at 3½ per cent. per annum?

8. If 17⅗ ells each containing 5 quarters, cost £6 17s., how much will 18 yards cost?

9. A bankrupt's stock was sold for £520 10s., at a loss of 17 per cent. on the cost price: had it been sold in the ordinary course of trade it would have realized a profit of 20 per cent.; how much was it sold at below the trade price?

10. If the quartern loaf be sold for 7½d. when wheat is 47s. per quarter, what would be its price when the price of wheat is 65s.?

11. A parcel of goods bought for £18 was sold 4 months afterwards for £25; what was the gain per cent per annum?

12. If 8 oz. of bread are sold for 6d. when wheat is £15 a load, what should be the price of wheat, when 12 oz. are sold for 4d.?

13. An annuity of £50 is put out to interest immediately after each payment; what will it amount to in 7 years, allowing 5 per cent. simple interest?

14. A person invests £1,000 in 3 per cent. stock at 88⅜, what will be the amount of his half-yearly dividends?

15. What annual income will arise from two investments, the one of £1,472 at 3 per cent., and the other of £2,000 at 4¼ per cent., deducting income tax at 7d. in the pound?

16. What weight ought to be carried 25¾ miles for the same sum for which 3 cwt. are carried 40 miles?

17. What is the present value of £75, due 17 months hence, at 4 per cent.?

18. A man can reap 345$\frac{5}{7}$ square yards in an hour; how long will seven such labourers take to reap 6 acres?

19. A corn-factor buys 2 quarters of wheat at 39s. per quarter, and 7 bushels of a superior quality at 6s. per bushel; at what rate must he sell the mixture so as to gain £1 by the transaction?

20. At what rate per cent. will a person receive interest who invests his capital in the 3 per cents. when they are at 91?

21. A bequest of £468 is made to each of three persons: the first being a son of the testator pays a duty of 1 per cent; the second being a brother pays 3 per cent.; and the third, who is not related, pays 10 per cent.: what sum does each receive?

22. A person transfers £1,000 stock from the 4 per cents. at 90 to the 3 per cents. at 72; find how much of the latter stock he will hold.

23. At what price must an article be bought, that, being sold for £3 10s. 6d., 13 per cent. may be cleared?

24. What principal lent at simple interest on the 1st of January at 5½ per cent. would amount to £1,000 on the 29th September in the same year?

25. A bankrupt owes £1,537 3s. 4d. but can pay only £960 14s. 7d.; what will be the dividend? How much will one of the creditors receive whose debt is £276 11s. 6d.?

26. Reduce 2,746 American dollars 30 cents. to British money, exchange being at 4s. 3⅛d. British per dollar.

27. What sum must be put out at 4 per cent. compound interest to amount to £1,000 in 5 years?

28. The divisible receipts of a railway company for one year are £437,500, and there are 250,000 shares of £21 each; what is the dividend on each share, and what is the rate per cent. of the company's profits?

29. What annual income will arise from the investment of £1,800 in the 3¼ per cents. when they are at 87⅛?

30. I borrow £130 on the 5th of March and pay back £132 10s. 6d. on the 18th of October; what rate per cent. per annum of interest have I paid

31. A and B rent a field for £35 a year: A puts in 6 horses for the whole year; B, 5 horses for 11 months, and 3 more for 5 months: how much should each contribute towards the rent?

32. If 12 men build a wall 60 feet long, 4 thick, and 20 high in 24 days, working 12 hours a day, how many must be employed to build a wall 100 feet long, 3 thick, and 12 high in 18 days, working 8 hours a day?

33. A commission agent offers to take either 2¼ per cent. on the money passing through his hands, or ⅓ of the net profit of his sales. The sales amount to £300,000, of which 35 per cent. produce no profit, but the remainder realise a profit of 10⅛ per cent. What difference will it make to the agent which of his offers his customers accept?

34. In what time will the sun move through 50·1 seconds when it describes 360 degrees in 365 days 6 hours 9 minutes 9·6 seconds, the motion being supposed uniform?

35. In what time will £645 15s. amount to £960 11s. 0¾d. at 4⅛ per cent. per annum?

36. At what rate will any sum of money double itself in 4½ years?

37. The total number of degrees in the five angles of a pentagon is 540; find the angles of a pentagon which shall be to one another as 2, 3, 4, 5, and 6.

38. In what time will any sum of money double itself at $3\frac{1}{4}$ per cent. simple interest.

39. In discharging a debt of £200 due a year hence, allowing 5 per cent. simple interest, why ought I to pay more than £190, and how much more?

40. A tenant pays corn rent of 30 quarters of wheat and 12 of barley, Winchester measure; what is the value of his rent, wheat being at 60s. and barley at 64s. a quarter imperial measure, supposing a Winchester bushel to be to an imperial bushel as 32 to 33?

41. The mint price of gold is £3 17s. $10\frac{1}{2}$d. per ounce; what is the smallest number of exact ounces that can be coined into an exact number of sovereigns?

42. A tradesman marks his goods at two prices, one for ready money and the other for 6 months' credit; what fixed ratio ought the two prices to bear to one another, allowing 5 per cent. per annum simple interest? and what should be the credit price of an article in his shop marked for ready money at £120 10s.?

43. The population of London in 1801 was 958,863; of Edinburgh, 81,404. In 1851, London contained £2,362,236 inhabitants; and Edinburgh, 191,221. What was the per centage of increase in each case?

44. A man pays taxes amounting to 10 per cent. on his income, but after these are paid he has £1,250 per annum to spend; what is his income?

45. At 1s. $7\frac{1}{2}$d. per day, how long will a man be in saving £10.

46. Out of an allowance of £50 a year, how much will be saved after spending 2s. $6\frac{1}{2}$d. a day?

47. A man spends 19s. 6d. a day, and saves 10 guineas every quarter; what is his income ?

48. What sum of money invested at $4\frac{1}{2}$ per cent. will amount to £1000 in a year and a half?

49. At what rate per cent. is the profit which a stationer makes who sells a book at a reduction of 2d. in the shilling on the published price, and purchases it at 25 per cent. discount?

50. If $\frac{3}{7}$ of 24 lbs. cost $\frac{19}{8}$ of 5s., what quantity can be purchased for £$203\frac{5}{8}$.

51. A pond contains 262,080 tons of water, and from a stream issuing from it 162 tons flow per hour: it is also supplied by a stream which brings 18 tons per hour; in what time will it be emptied?

52. If 40 men can do a piece of work in 24 days, how many men will it take to finish a piece six times as great in ⅖ of the time?

53. If 6 men and 9 boys can finish a certain work in 108 days, working together, in what time will one man and a boy do it when working together, suppose the boy to do ⅖ of the work of a man?

54. What is the discount on £257 8s. 8¼d. if paid 210 days before it is due, interest being at 4½ per cent.?

55. If two trains meet, the one 150 yards long, moving at the rate of 50 miles an hour, and the other 240 yards long, moving at the rate of 40 miles an hour, how long will they be in passing each other?

56. Two gas-meters communicate with each other by a pipe and stopcock; the one contains 4,056 cubic inches, and the other 8,096 cubic inches, of gas at the maximum density: a quantity equal to 5½ per cent. of the former is ejected; what per centage of the second must be let off to supply the deficiency?

57. A pole, one-third of which is in the ground, projects a shadow of 3·5 feet; what is its whole length if a tower 100 feet high has at the same moment a shadow of 23⅓ yards?

58. If 12 days' labour be worth £1 11s. 6d., how much will be required to pay the wages of 4 workmen, who have laboured 6, 9, 10½ and 11 days respectively?

59. A bankrupt owes £900 to his three creditors, and his whole property amounts to £675; two of his creditors claim £122 and £376 respectively, what dividend will the other receive?

60. A watch set accurately at 1 p.m. indicates 10 minutes to 7 at 7 o'clock; what will be the true time when the hands point to 7?

61. If 40 oz. of silk can be spun into a thread 2 furlongs in length, what weight of silk would supply a thread sufficient to reach to the sun, whose distance from us is 95,000,000 of miles?

62. The attraction of bodies varies directly as their density and magnitude, and inversely as the squares of their distances. Compare the effect which two globular bodies will have upon a third body, the density of the first being 5, its diameter 96, and its distance from the third body 500,000 miles; while the density of the second is 11, its diameter 80, and its distance 250,000 miles.

63. If the interest of £725 10s. for a certain time at 3 per cent. be £21 15s. 3⅜d., at what rate per cent. will the same sum amount to £785 8s. 0¾d. in the same time?

64. If 15 cwts. 1 qr. 2 lbs. of sugar be bought for £52 0s. 3d., at what price per lb. should it sell to gain 15 per cent.?

INVOLUTION AND EVOLUTION.

407. The products formed by multiplying the same number into itself are called the *Powers* of the number.

Thus the product of three sevens, or 7 × 7 × 7, is called the *Third Power* of 7; the product of five sevens, or 7 × 7 × 7 × 7 × 7, is called the *Fifth Power* of 7. These two results are written 7^3 and 7^5.

The figure which represents the power to which a number is raised is called the *Exponent*, or *Index;* thus, in 7^3 and 7^5 the 3 and 5 are exponents, the one showing that 7 is to be raised to the third, and the other that it is to be raised to the fifth power.

The second power of a number is called its *Square*, and the third its *Cube.**

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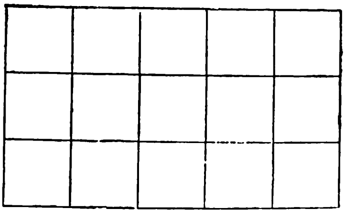
* These terms are taken from geometry, and their use may be accounted for by the fact that when the manner of forming the second and third powers of numbers was first investigated it was with a view to the measurement of surfaces and solids. There are two simple propositions which will justify this connexion of arithmetic with geometry, and which are self-evident as soon as stated.

I. *If any two numbers represent the units of length in the adjacent sides of a rectangular parallelogram, their product will represent the units of surface in the parallelogram itself.*

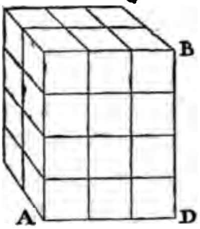


Thus: because one side contains 5 units of a certain length and the other side 3 units of the same length, the whole parallelogram contains 5 × 3, or 15 square surfaces, each having the unit of length for one side.

Corollary. — Whenever the two sides of the parallelogram are equal and the figure is a square, the number representing the area is the second power of the number representing the length of the side. Hence the second power is often called a *square.*

408. When we find the result of multiplying a number by itself a certain number of times, we are said to *involve* that number to the given power, and the operation is called *Involution.* Thus 5 × 5 × 5 = 125. The number 5 has here been *involved* to the third power.

II. *If any three numbers represent the units of length of the three dimensions of a rectangular solid, the product of those three numbers represents the units of solidity in the body itself.*



Thus: because the height BD is represented by 4, the length AD by 3, and the width BC by 2, the whole mass contains 4 × 3 × 2, or 24 regular solids, each having the given unit of length for each of its dimensions.

Corollary,—Whenever the solid is equal in all its dimensions and is therefore a cube, the number representing the solidity is the third power of the number which represents the length of each dimension. Hence the third power of a number is often called a *cube.*

In old books several other terms, borrowed from mensuration, were employed in arithmetic. Thus, in the 8th Book of Euclid the product of any two numbers is called a *plane* number, because it is a number which might represent the extent of a surface; and the product of any three numbers is called a *solid* number, because it might represent the magnitude of a rectangular mass. But these expressions have become obsolete, and as there is evidently nothing in geometry analogous to the fourth, fifth, or any higher powers of a number.the only geometrical terms employed in modern arithmetic are square and cube.

It follows, from these considerations, that any theorems in geometry which show the relative magnitude of rectangles and squares formed upon certain lines, will illustrate that part of arithmetic which relates to the formation of the products of numbers and their second powers. For example, the 2nd Book of Euclid investigates the conditions under squares and rectangles and the relations of the lines and parts of lines on which they are constructed: to every proposition in that book, therefore, there is a corresponding proposition referring to the products of numbers. In like manner, the truths of solid geometry will point to some analogous truths in relation to the products of numbers and their third powers. But it must be remembered, that although in these cases arithmetical and geometrical theorems *illustrate* one another, we must not attempt to prove the one by proving the other, The evidence of a geometrical proposition lies in certain axioms and elementary truths, which are founded on our idea of space; but the proof of all arithmetical propositions rests on axioms and notions of number only. Hence the propositions in the 2nd Book of Euclid cannot be proved numerically or algebraically, as some students suppose; they must be demonstrated as pure geometry, and the idea of number should be as far as possible excluded. In like manner, there can be no geometrical proof of the rules for the involution of numbers; and although many arithmetical propositions will be found analogous to familiar theorems in geometry, their evidence depends on the principles of arithmetic alone.

But when a certain power of a number is given and we are required to find that number of which it is the power, we are said to *extract* or *evolve* the root, and the process is called *Evolution.* Thus to find that 125 is made up of 5 × 5 × 5 is to evolve the 5, and 5 is called the third or cube *root* of 125.

TABLE OF POWERS.

	1,	2,	3,	4,	5,	6,	7,	8,	9.
Second power	1	4,	9,	16,	25,	36,	49,	64,	81.
Third power	1,	8,	27,	64,	125,	216,	343,	512,	729.
Fourth power	1,	16,	81,	256,	625,	1296,	2401,	4096,	6561.
Fifth power	1,	32,	243,	1024,	3125,	7776,	16807,	32768,	59049.

409. Every number may be a *root,* because it may be involved or multiplied by itself as often as we please; but very few numbers are *powers* of other integer numbers.

410. *Signs.* The character $\sqrt{\ }$ is the sign of Evolution;

Thus, $\sqrt{64}$ = the square root or second root of 64.

$\sqrt[3]{125}$ = the cube root or third root of 125.

$\sqrt[5]{64}$ = the fifth root of 64.

A number is called a square number when it has an integer number for its square root.

Thus: 25 is a square number, because $\sqrt{25} = 5$, and $\sqrt{25}$ is a *rational* quantity,*

A number whose root cannot exactly be ascertained is not a square number, and its root is called an *irrational quantity* or *surd.* Thus: 12 is not a square number, because no number can be found equal to $\sqrt{12}$, and the quantity $\sqrt{12}$ is a surd or irrational quantity.

Observation.—It is evident that since a surd is a number not expressible in figures, no surd multiplied or divided by any figures, whether integral or fractional, can give a rational number as the product or quotient.

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* *Rational* is derived from *ratio,* used in the sense explained on page 187. Numbers are *rational* when their *ratio* or relation to other numbers can be exactly represented. Hence every number which can be expressed in figures is rational. But $\sqrt{12}$ or $\sqrt{3}$ is irrational, because no figures whatever could represent the ratio here intended. irrational or incommensurable quantities were called by the Greeks ἄλογα, because their λογος (*logos*), or ratio, could not be stated.

SECTION I.—INVOLUTION TO THE SECOND POWER, OR FORMATION OF THE SQUARES OF NUMBERS.

411. All the operations of Involution rest upon an axiom in Multiplication (72). *Multiplication is always effected between two factors, if each of the parts of the one is multiplied by each of the parts of the other, and the sum of these products be taken. Hence—*

412. *If a number be divided into any two parts, the square of the number is equal to the product of the whole and one part, plus the product of the whole and the other part.*

Demonstrative Example.—Because $18 = 10 + 8$;

$$18 \times 18 = (18 \times 10) + (18 + 8).$$

General Formula.—If $a = b + c$, $aa = ab + ac$.

413. *If a number be divided into two parts, the product of the whole and one part is equal to the square of that part, plus the product of the two parts.*

Demonstrative Example.—Because $18 = 10 + 8$;

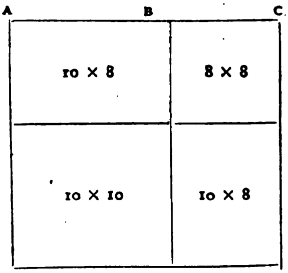
$$\therefore 18 \times 10 = (10 \times 10) + (10 \times 8);$$

$$\text{and } 18 \times 8 = (8 \times 8) + (10 \times 8).$$

General Formula.—If $a = b + c$, $ab = b^2 + bc$, and $ac = c^2 + bc$.

414. *If a number consist of two parts, the square of that number consists of the squares of those two parts, together with twice their product.*

Geometrical Illustration.—Let the line AC be divided into two parts, of which AB = 10 and BC = 8, it is evident that the whole area of the square described upon that line is made up of the four parts 10^2, 8^2, and two of (10×8).



Demonstrative Example.—For by (412) 18×18 was resolved into $(18 \times 10) + (18 \times 8)$.

And by (413) 18×10 was resolved into $(10 \times 10) + (10 \times 8)$, and 18×8 into $(8 \times 8) + (8 \times 10)$.

Therefore $18 \times 18 = (10 \times 10) + (8 \times 8) + (10 \times 8) + (10 \times 8)$
$= 10^2 + 8^2 \times 2\,(10 \times 8)$.

General Formula.—Whenever $a = b + c$; then $a^2 = b^2 + c^2 + 2bc$.

EXERCISE CXI.

☞ Divide each of the following numhers into two parts, and show how they illustrate propositions 412 and 413.

11, 15, 8, 24, 50, 19, 100, 26, 327, 48, 59, 25, 46, 72.

EXERCISE CXII.

☞ Divide each of the following numbers into two parts, and show how they illustrate proposition 414.

15, 18, 24. 65, 23·5, 49, 10·8, 796, 235, 47, 32, 56, 71.

415. *Corollary.—The square of any number is equal to four times the square of half that number.*

Demonstrative Example.—The product of two equal numbers is the square of either of them, and because $18 = 9 + 9$, then by (414)

$18 \times 18 = (9 \times 9) + (9 \times 9) + 2\,(9 \times 9)$, or $18^2 = 4 \times 9^2$.

General Formula.—If $a = 2b$, then $a^2 = 4b^2$.

EXERCISE CXIII.

☞ Verify each of the last four propositions by twelve different numerical examples of your own selection.

416. *If a number be divided into three parts, the square of the whole is made up of the square of the first; twice the product of the first into the sum of the second and third; the square of the second; twice the product of the second and third; and the square of the third.*

Demonstrative Example.—Divide 15 into 7, 5, and 3.
Consider the 5 and 3 as one part and indicate it thus, $\overline{5+3}$.
Then (414) 15^2 or $(7 + 5 + 3)^2 = 7^2 + 2\,(7 \times \overline{5 + 3}) + (5 + 3)^2$.
But by (414) $(5 + 3)^2 = 5^2 + 2\,(5 \times 3) + 3^2$.
Hence $15^2 = 7^2 + (2 \times 7 \times \overline{5 + 3}) + 5^2 + (2 \times 5 \times 3) + 3^2$.
Or $225 = 49 + 112 + 25 + 30 + 9$.

General Formula.—Let $a = b + c + d$;
then $a^2 = b^2 + 2b\,(c + d) + c^2 + 2cd + d^2$.

Exercise CXIV.

Divide each of the following numbers into three parts, and show how they illustrate this proposition :—

6, 11, 18, 24, 39, 58, 111, 416, 72, 344, 718, 654, 419,

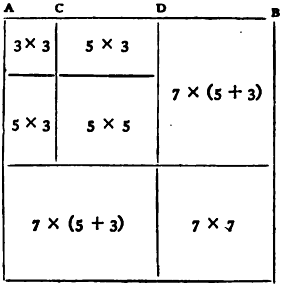
417. *If a number be divided into any number of parts, the square of the whole is made up of the squares of each of the parts, together with twice the product of each part into the sum of all the succeeding parts.*

Demonstrative Example.—Because $12 = 3 + 9$;

$\therefore$ by (414) $12^2 = 3^2 + (2 \times 3 \times 9) + 9^2$.
But $9 = 2 + 7$, and $9^2 = 2^2 + (2 \times 2 \times 7) + 7^2$.
Wherefore $12^2 = 3^2 + (2 \times 3 \times 9) + 2^2 + (2 \times 2 \times 7) + 7^2$.
But $7 = 4 + 3$ $\therefore 7^2 = 4^2 + (2 \times 4 \times 3) + 3^2$.
Wherefore $12^2 = 3^2 + (2 \times 3 \times 9) + 2^2 + (2 \times 2 \times 7) + 4^2 + (2 \times 4 \times 3) + 3^2$

General Formula.—If $a = b + c + d + e + f$;

Then $a^2 = b^2 + 2b(c + d + e + f) + c^2 + 2c(d + e + f) + d^2 + 2d(e + f) + e^2 + 2ef + f^2$.



Geometrical Illustration.—Let the length of the line AB be 15, and let it be divided into three parts, of which DB measures 7, CD 5, and AC 3, the area of the square described upon it is evidently made up of seven portions, whose dimensions are those indicated by the numbers in the diagram.

And $15^2 = 7^2 + 2 \times 7 \times (5+3) + 5^2 + 2 \times 5 \times 3 + 3^2$.

EXERCISE CXV.

☞ Verify by actual calculation each of the following assertions :—

Because $20 = 9 + 6 + 3 + 2$, $\therefore 20^2 = 9^2 + (2 \times 9 \times 11) + 6^2 + (2 \times 6 \times 5) + 3^2 + (2 \times 3 \times 2) + 2^2$.

$2738^2 = 2000^2 + (2 \times 2000 \times 738) + 700^2 + (2 \times 700 \times 38) + 30^2 + (2 \times 30 \times 8) + 8^2$.

EXERCISE CXVI.

☞ Divide each of the following numbers into not less than four parts, and show how its square is made up :—

4, 13, 25, 41, 38, 65, 312, 417, 982, 645, 123, 444, 7098.

418. It follows from the laws of our system of notation* that—

418. *The square of any number cannot contain more than twice as many figures as are contained in the number itself, nor less than twice as many, minus one.*

Demonstrative Example.—The highest number which can be written with three digits only is 999. The square of this number must be less than the square of 1000. But $1000^2 = 1000000$, and this is the smallest number which extends to the seventh place. Wherefore the square of 999 cannot contain more than six digits. Again, the lowest number which requires three figures to express it is 100, and $100^2 = 10000$. But this number extends to the fifth place. Wherefore the square of any number consisting of three digits cannot contain more than six nor less than five digits.

In the same manner it might be proved that—

The square of a number of 2 places cannot have more than 4 or less than 3 digits.

The square of a number of 4 places cannot have more than 8 or less than 7 digits.

The square of a number of 5 places cannot have more than 10 or less than 9 digits.

And, generally, if a number contain n figures its square contains not more than $2n$ and not less than $2n - 1$.

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* The student should notice the phraseology here employed. The fact here stated is not a fundamental principle of the Science of Arithmetic, but one of the consequences of our having adopted the system of notation described in (15.)

** SECTION II.—THEORY OF NUMERICAL SQUARES AND PRODUCTS (*continued*).

419. The theorems in this section are intended still further to illustrate the nature of a square number, and the manner in which it may be formed. But they are not necessary for the comprehension of the ordinary rule for extracting the square root, and may be passed over by the student who is reading this book for the first time.

420. *The square of the product of two or more numbers is the same as the product of their squares.*

Demonstrative Example.—$(5 \times 8)^2 = 5^2 \times 8^2 = 1600$.

For by (65) factors may be multiplied in any order; therefore $(5 \times 8)\,(5 \times 8)$, or $5 \times 8 \times 5 \times 8 = 5 \times 5 \times 8 \times 8$, or $5^2 \times 8^2$.

General Formula.—$(ab)^2 = a^2b^2$, because $abab = aabb$.

421. *Corollary.—If one number measures another its square measures the square of that other.*

Demonstrative Example.—Because 3 is a measure of 12, 3^2 is a measure of 12^2; for $3 \times 4 = 12$, therefore (420) $3^2 \times 4^2 = 12^2$, and 3^2 is a measure of 12^2.

General Formula.—If a be a measure of b, a^2 is a measure of b^2.

For let a be contained in b m times, then $am = b$, and $a^2m^2 = b^2$. Wherefore a^2 is a measure of b^2.

EXERCISE CXVII.

☞ Resolve each of the following numbers into two or more factors, and prove that the square of the products equals the product of the squares:—

6, 15, 27, 48, 50, 240, 168, 144, 36, 25, 72.

422. *The product of the sum and difference of any two numbers equals the difference of their squares.*

Example.—Let the numbers be 12 and 8; their sum is 20, or $(12 + 8)$, and their difference 4, or $(12 - 8)$.

$(12 + 8) \times (12 - 8) = 12^2 - 8^2$; *i.e.*, $20 \times 4 = 144 - 64$.

General Formula.—$(a + b) \times (a - b) = a^2 - b^2$.

423. *Corollary I.—If there be any two numbers whose difference is one, the difference of their squares equals their sum.*

Example.—Take the series of numbers—

1, 2, 3, 4, 5, 6, 7, 8, 9, &c.

The difference between the square of 1 and the square of 2 is 1 + 2; the difference between the square of 2 and the square of 3 is 2 + 3; between 4^2 and $5^2 = 4 + 5$, and so on.

General Formula.—By (414), $(a + b)^2 = a^2 + 2ab + b^2$.

$$\therefore (a + 1)^2 = a^2 + 2a + 1^2.$$
$$\therefore (a + 1)^2 \text{ exceeds } a^2 \text{ by } 2a + 1.$$

Hence the difference between the squares of any two consecutive numbers = twice the less + 1, and this is necessarily the sum of the two numbers. Thus 378^2 exceeds 377^2 by $2 \times 377 + 1$, or 755. But this is 378 + 377.

424. *Corollary II.—If there be two fractions which added together make one, the difference of their squares is the difference of the fractions themselves.*

For in this case the product of the sum and difference = the difference, the sum being unity.

Example.—Because $\frac{3}{4} + \frac{2}{8} = 1$, $(\frac{3}{4})^2 - \frac{2}{8})^2 = \frac{3}{4} - \frac{2}{8}$.

Or $\frac{9}{16} - \frac{4}{64} = \frac{36}{64} - \frac{4}{64} = \frac{32}{64} = \frac{1}{2}$, which is the difference between $\frac{3}{4}$ and $\frac{2}{8}$.

In no other case can the difference between the squares of two fractions be the same as between the fractions themselves.

EXERCISE CXVIII.

☞ Take the sum and difference of the following pairs of numbers, and show that their product equals the difference of their squares:—

1. 5 and 9; 28 and 10; 11 and 12; 14 and 17; 15 and 9.
2. 6 and 7; 8 and 15; 23 and 24; 100 and 150; 65 and 85.
3. $\frac{1}{4}$ and $\frac{1}{5}$; $\frac{2}{7}$ and $\frac{4}{9}$; $\frac{5}{12}$ and $\frac{14}{24}$; ·47 and ·01; 1·25 and ·16.

425. *The square of the difference between any two numbers equals the sum of their squares, minus twice their product.*

Example.—$(12 - 7)^2 = (12 - 7)(12 - 7)$.

Now $(12 - 7)(12 - 7) = 12^2 - 2(12 \times 7) + 7^2$.

$$i.e., \; 5 \times 5 = 144 + 49 - 168 = 25.$$

General Formula.—$(a - b)(a - b) = a^2 - 2ab + b^2$.

EXERCISE CXIX.

☞ Take the difference between the two numbers of each of the following pairs, and show how this truth may be illustrated :—

1. 12 and 4; 17 and 6; 2 and 3; 14 and 17.
2. 20 and 14; 53 and 55; 100 and 9; 27 and 13.
3. $\frac{1}{3}$ and $\frac{1}{4}$; $\frac{3}{8}$ and $\frac{3}{7}$; $\frac{4}{11}$ and $\frac{6}{17}$; $\frac{16}{18}$ and $\frac{3}{8}$.
4. ·25 and 1·2; 408 and 5·6; 11·1 and 111; 18·3 and ·002.

426. From propositions (414) and (425) we find that—

The square of the sum of two numbers exceeds the sum of their squares by twice their product.

The square of the difference of two numbers is less than the sum of their squares by twice their product.

We have, therefore, in all cases in which two numbers are concerned, three magnitudes—

$$\begin{array}{ccc} (7 + 5)^2 & 7^2 + 5^2 & (7 - 5)^2 \\ (a + b)^2 & a^2 + b^2 & (a - b)^2 \end{array}$$

of which the first exceeds the second as much as the second exceeds the third. Wherefore by (54) the sum of the first and third equals twice the second, or $(a + b)^2 + (a - b)^2 = 2\,(a^2 + b^2)$, or *the square of the sum of two numbers, together with the square of their difference, make up twice the sum of their squares.* Hence also the following corollary may be deduced.

427. *Corollary II.—The square of the sum of two numbers exceeds the square of their difference by four times their product.*[*]

Demonstrative Example.—For $(10 + 3)^2$ exceeding $10^2 + 3^2$ by twice the product of 10 and 3, and $10^2 + 3^2$ exceeding $(10 - 3)^2$ by twice their product also, therefore $(10 + 3)^2$ must exceed $(10 - 3)^2$ by four times their product.

General Formula.—$(a + b)^2 - (a - b)^2 = 4ab$.

EXERCISE CXX.

☞ Verify the last two corollaries by taking the squares of the sum and difference of the following pairs of numbers :—

1. 7 and 12; 4 and 9; 18 and 7; 17 and 13.
2. 16 and 20; 14 and 16; 24 and 20; 15 and 25.
3. 10·5 and 8·5; 12·1 and 50·9; 1·6 and 1·8; 40 and 45.

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[*] This proposition is analogous to Euclid's 8th proposition, Book ii.

428. *Corollary III.—Four times the square of half the sum of any two numbers exceeds four times the square of half their difference by four times their product.*

Demonstrative Example.—Because by (427)

$$(10 + 6)^2 - (10 - 6)^2 = 4 \times 10 \times 6;$$

And by (415) $(10+6)^2 = 4\left(\frac{10+6}{2}\right)^2$ and $(10-6)^2 = 4\left(\frac{10-6}{2}\right)^2$

$$\therefore\ 4\left(\frac{10+6}{2}\right)^2 - 4\left(\frac{10-6}{2}\right)^2 = 4 \times 10 \times 6.$$

General Formula.—$4\left(\frac{a+b}{2}\right)^2 - 4\left(\frac{a-b}{2}\right)^2 = 4ab.$

If now we take a fourth part of every one of these quantities we have the following result:—

429. *Corollary IV.—The square of half the sum of two numbers exceeds the product of those numbers by the square of half their difference.*[*]

Example.—$\left(\frac{10+6}{2}\right)^2 - \left(\frac{10-6}{2}\right)^2 = 10 \times 6.$

$10^2 = 11 \times 9 + 1^2$
$10^2 = 12 \times 8 + 2^2$
$10^2 = 13 \times 7 + 3^2$
$10^2 = 14 \times 6 + 4^2$
$10^2 = 15 \times 5 + 5^2$
$10^2 = 16 \times 4 + 6^2$
$10^2 = 17 \times 3 + 7^2$
$10^2 = 18 \times 2 + 8^2$
$10^2 = 19 \times 1 + 9^2$

It will be seen here, in how many ways 10^2 or 100 may be made up.

Because $10 + 10 = 20$, the product of any two numbers whose sum is 20, added to the square of half their difference, will equal 10^2.

General Formula.—$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a-b}{2}\right)^2.$

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[*] The ordinary rule for squaring a number by mental arithmetic is founded on this principle. It is, "Add the lower unit to the upper, multiply by the tens, and add the square of the unit." Thus 34 × 34, by adding the lower unit to the upper, becomes 38 × 30, or 1140. But this product of two numbers is less than the square of half their sum (34), by the square of half their difference (4). Hence $38 \times 30 + 4^2$, or $1140 + 16$, or $1156 = 34^2$.

When the unit is greater than 5 it will be more convenient to put the rule into another form:—"Take from the one unit a number which will make the other an even *ten*; multiply these two numbers together, and add the square of the number which was subtracted," *e.g.*, 67^2. Add 3 to one 67 and take it from the other, then 70 × 64, or 4480, is the product to be obtained. But this product of 70 × 64 is less than the square of 67, or half their sum, by the square of 3, or half their difference, and $70 \times 64 + 3^2$, or $4480 + 9$, or $4489 = 67^2$.

By the other method $74 \times 60 + 7^2$, or $4440 + 49 = 4489 = 67^2$.

Corollary IV. is analogous to the 5th proposition of Euclid, Book ii.

EXERCISE CXXI.

☞ Make up the square of each of the following numbers in four different ways, by the help of the last corollary :—

1. 7 ; 12 ; 15 ; 20.
2. 5 ; 16 ; 8 ; 24.
3. 14·4 ; 26 ; 30·5 ; 84·7.
4. $\frac{3}{4}$; $7\frac{1}{3}$; $16\frac{4}{5}$; $11\frac{1}{8}$.
5. $16\frac{1}{9}$ $43\frac{2}{3}$ $29\frac{1}{5}$; $104\frac{2}{7}$

430. The truth of the following propositions may easily be inferred from the foregoing. They should be verified in each case by the use of numerical examples.

1. $\left(\frac{A+B}{2}\right)^2 + \left(\frac{A-B}{2}\right)^2 = \frac{A^2 \times B^2}{2}$

   or $\left(\frac{7+5}{2}\right)^2 + \left(\frac{7-5}{2}\right)^2 = \frac{7^2+5^2}{2}$

2. $(A + B)^2 + (A + B)(A - B) = 2A(A + B)$
   or $(7 + 5)^2 + (7 + 5) \times (7 - 5) = 2 \times 7(7 + 5)$.
3. $(A - B)^2 + (A + B)(A - B) = 2A(A - B)$
   or $(7 - 5)^2 + (7 + 5) \times (7 - 5(= 2 \times 7(7 - 5)$.
4. $(A + B)^2 - (A + B)(A - B) = 2B(A + B)$
   or $(7 + 5)^2 - (7 + 5) \times (7 - 5) = 2 \times 5(7 + 5)$.
5. $(A + B)(A - B) - (A - B)^2 = 2B(A - B)$
   or $(7 + 5) \times (7 - 5) - (7 - 5)^2 = 2 \times 5(7 - 5)$.
6. $(A + B)B + \left(\frac{A}{2}\right)^2 = \left(\left(\frac{A}{2}\right) + B\right)^2$

   or $(6 + 4)\,4 + \left(\frac{6}{2}\right)^2 = \left(\frac{6}{2}\right)^2 \cdot \left(\frac{6}{2}+4\right)^2$
7. $(A + B)^2 + B^2 = 2(A + B)B + A^2$
   or $(7 + 5)^2 + 5^2 = 2(7 + 5)\,5 + 7^2$.
8. $(A + B)^2 + A^2 = 2(A + B)A + B^2$
   or $(7 + 5)^2 + 7^2 = 2(7 + 5)\,7 + 5^2$.

EXERCISE CXXII.

☞ (a): Express each of the foregoing propositions in words, and give six numerical illustrations of each.

(b). Make four numerical illustrations of each of the propositions in this section.

SECTION III.—EVOLUTION FROM THE SECOND POWER, OR EXTRACTION OF THE SQUARE ROOT OF NUMBERS.

431. As in Division and other parts of Arithmetic, so here, it is necessary to break up every sum into such smaller sums as shall come within the range of our tables; and in fact, to do to a number part by part what we wish to do to the whole. In all former rules, however, we have found that whatever was done to the parts successively was done to the whole; but in extracting the roots of numbers this is not the case, for, *by finding the roots of the several parts which compose a number, we do not find the root of the whole number;* if it were so, the squares of these several parts added together would equal the square of the whole number, which by (414) is impossible.

432. *If there be a number whose root when extracted will consist of two parts, that number must contain not only the squares of those two parts, but twice their product also.*

This is only another form of the truth stated in (414), it is here put in the form adapted to the inverse process of extracting a root, and needs no demonstration.

433. *Example I.*—Find the square root of 25.

$$\begin{array}{l} 25 \\ \underline{9} = 3^2 \\ 16 \\ \underline{12} = 2\,(2 \times 3) \\ 4 \\ \underline{4} = 2^2 \\ \cdot\cdot \end{array}$$

We first choose a number, 3, whose square is certainly contained in 25; on taking this away we observe that 16 remain.

If we were to take the square root of this remainder, which is 4, and add it to the 3, we should be clearly wrong, for the square root of $(9+16)$ is not $(3+4)$ but since a number, if it contains the square of the sum of two others, must contain the sum of their squares *and twice their product* (414), we must be able to take from the 16, not only the square of the new part, but also twice the product of it and the other part.

Now if we choose the number 2, we observe that its square (4), and twice the product of itself and the first found number, 3, will make up 16, or that $2^2 + 2\,(2 \times 3) = 16$.

Hence the number 25 has had taken from it in succession—the square of 3, twice the product of 3 and 2, and the square of 2.

Therefore 25 contains the square of (3 + 2),

for $(3 + 2)^2 = 3^2 + (2 \times 3 \times 2) + 2^2$.

434. *Example II.*—Extract the second root of 1225.

$$\begin{array}{l} 1225 \\ \underline{900} = 30^2 \\ 325 \\ \underline{325} = 2 \times 30 \times 5 + 5^2 \\ \cdots \end{array}$$

We observe that there are 12 hundreds here. Now the nearest number of hundreds whose root can be easily ascertained, is 900, which is the square of 30. But the root of 1225 must be greater than 30, because 325 remain; let the other part of the root be a, then the root is $(30 + a)$; if so, then $1225 = 30^2 + (2 \times 30a) + a^2$. As 30^2 has already been taken away, the remaining 325 must contain twice the product of 30 into the new part, together with the square of the new part, $(2 \times 30)\, a + a^2$, or $(2 \times 30 + a)\, a$. If therefore we double the 30, choose a new part, 5, and add to it, and then multiply the 65 by the new part, this product, 325, will equal $(2 \times 30 \times 5) + 5^2$, and as this is the same as the remainder, the whole number $1225 = (30 + 5)^2$.

435. ***If we divide any number into portions by pointing off every second figure, beginning with the unit, the number of the points thus made will show how many figures are contained in the root.***

Demonstrative Example.—The root of 56218 will contain three figures for by (418) the first portion, 50000, will have for its root a number of the third place, the second portion, 6200, will have a number of the second place, and the third, 18, will have a unit for its root.

Hence, to find what will be the place of the first figure in the root it is usual to place a point over the first, the third, the fifth, and every alternate place in the number. Thus $\dot{5}6\dot{2}1\dot{8}$.

436. ***If the square root of a number consist of several parts, the number itself contains the square of each part, together with twice the product of that part into the sum of all the preceding parts.***

This is only the same proposition as (417) expressed in a form adapted for evolution, and needs no demonstration.

437. *Example III.*—Extract the square root of 613089.

$$\begin{array}{rl} & 6\dot{1}3\dot{0}8\dot{9}\ (700 + 80 + 3 = 783) \\ 700^2 = & \underline{490000} \\ & 123089 \\ (1400 + 80)\,80 = & \underline{118400} = 2 \times 700 \times 80 + 80^2 \\ & 4689 \\ (1560 + 3)\,3 = & \underline{4689} = 2 \times 3 \times (700 + 80) + 3^2 \\ & \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

On pointing this number, $6\dot{1}3\dot{0}8\dot{9}$, we first observe that there will be three figures in the root, *i. e.*, that it will consist of a number of hundreds; 610000 is therefore the first part of the number selected. Now because $\sqrt{61}$ is more than $\sqrt{49}$ and less than $\sqrt{64}$, the root of 61 tens of thousands must be more than 700 and less than 800. 700 is therefore the first part of the root, and on subtracting its square we find that 123089 remains. We next have to find how many tens are in the root. Whatever this number is, its square, together with twice the product of itself and 700, must be contained in the remainder, *i. e.*, $2 \times 700 \times$ the number of tens + the square of the number of tens; or if x be the number of tens $(1400 + 10x)\,10x$. Now on applying 14000 to the remainder, we see that 8 is the nearest quotient, we therefore take 80 as the second part of the root. After subtracting the square of this part, together with twice the product of itself and the former part, we find 4689 remaining. We have now to find the third part of the root, and by (436) this must be such a number that its square + twice the product of itself and the sum of the preceding parts shall be contained in 4689, *i. e.*, 2 (780 × the new number) + the square of the new number. But $2 \times 780 = 1560$, and if we apply this to 4689 we have the quotient 3. Wherefore $2 \times 780 \times 3 + 3^2$, or 1563×3, or 4689, should be contained in the remainder.

438. It was stated in (416) that $(a + b + c)^2 = a^2 + 2a\,(b + c) + b^2 + 2bc + c^2$. Now the three parts of the number we have here found are 3, 80, and 700, and we have taken away—

$$3^2 + (2 \times 3 \times 780) + 80^2 + (2 \times 700 \times 80) + 700^2.$$

$$\text{Hence} \quad \sqrt{613089} = 700 + 80 + 3 = 783.$$

In this case the number proposed is an exact square, but in most cases a remainder is left after the operation, and we are only able to find the *nearest* square root.

Extraction of the Square Root by Approximation.

439. *The square root of a fraction is always to be found by taking the square root of the numerator for a new numerator, and the square root of the denominator for a new denominator.*

Demonstrative Example.—

$$\text{Because } \frac{5}{7} \times \frac{5}{7}, \text{ or } \left(\frac{5}{7}\right)^2 = \frac{5^2}{7^2} = \frac{25}{49};$$

$$\text{Therefore the square root of } \frac{25}{49} \text{ is } \frac{\sqrt{25}}{\sqrt{49}} = \frac{5}{7}.$$

General Formula.— $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$

This principle is applied in several ways; *e.g.*,

440.—I. If the numerator and denominator are both multiplied by the denominator, the root of the fraction may always be so found as to give a rational denominator.

Let it be required to find the square root of $\frac{10}{7}$. Multiply both terms of the fraction by the denominator.

$$\text{Hence, because } \frac{10}{7} = \frac{10 \times 7}{7 \times 7};$$

$$\text{Therefore } \sqrt{\frac{10}{7}} = \sqrt{\frac{70}{49}} = \frac{\sqrt{70}}{7}.$$

But the nearest root of 70 is 8, therefore $\frac{8}{7}$ is the nearest result possible with the denominator 7.

441.—II. If any number be converted into a fractional form, with a rational number for its denominator, its root can be approximately found.

Let it be required to find the square root of 59, so that the answer shall not differ by so much as $\frac{1}{12}$ from the truth.

$$\text{Because } 59 = \frac{59 \times 12^2}{12^2};$$

$$\text{Therefore } \sqrt{59} = \sqrt{\frac{8496}{12^2}} = \frac{\sqrt{8496}}{12}.$$

But the nearest square root of 8496 is 92.

$$\text{Therefore } \sqrt{59} = \frac{92}{12} = 7\tfrac{2}{3} \text{ nearly.}$$

Again, let it be required to find the square root of 54 true to $\frac{1}{3}$.
Multiply and divide 54 by the square of 3.

$$\text{Hence, because } 54 = \frac{54 \times 3^2}{3^2};$$

$$\text{Therefore } \sqrt{54} = \sqrt{\frac{486}{9}} = \frac{\sqrt{486}}{3}.$$

But the nearest square root of 486 is 22.

$$\text{Hence } \sqrt{54} = \frac{22}{3} = 7\tfrac{1}{3} \text{ nearly.}$$

This answer does not differ so much as $\frac{1}{3}$ from the truth.
Required the root of $31\frac{4}{7}$ true to $\frac{1}{23}$.

$$31\tfrac{4}{7} = \frac{221}{7} = \frac{221 \times 23^2}{7 \times 23^2} = \frac{116909}{7} \div 23^2 = 16701\tfrac{2}{7} \div 23^2.$$

Neglecting the fractions in the numerator we have—

$$\sqrt{31\tfrac{4}{7}} = \sqrt{\frac{16701}{23^2}} = \frac{\sqrt{16701}}{23}.$$

But the nearest root of 16701 is 129.

$$\text{Hence } \sqrt{31\tfrac{4}{7}} = \frac{129}{23} = 5\tfrac{14}{23} \text{ nearly.}$$

This answer does not differ by $\frac{1}{23}$ from the truth.

442. It follows from these examples, that whenever accuracy is required to a given limit, say $\frac{1}{n}$, we must multiply the number by n^2, extract the root of this product, and divide the result by n.

443.—III. The roots of numbers which are expressed decimally can always be found readily on this method when the expression extends to an even number of decimal places.

Demonstrative Example.—Because (250) the unexpressed denominator of 791·82 is 100, the root of 791·82 = $\sqrt{79182} \div \sqrt{100}$, *i.e.*

$$\sqrt{791{\cdot}82} = \frac{\sqrt{79182}}{\sqrt{100}} = \frac{\sqrt{79182}}{10}.$$

Therefore if we find the root of 79182, neglecting the decimal, and then mark off the unit figure of the answer by a decimal point (*i.e.*, divide it by 10) we shall have the root of the fraction.*

---

* A fraction may have a rational square root, and yet appear to have none; *e.g.*, $\frac{75}{147}$; the roots of the numerator and denominator cannot be found; but if we divide both by 3 the fraction becomes $\frac{25}{49}$, the root of which is rational. Hence, before we extract the square root of a vulgar fraction, we must reduce it to its lowest name.

Had the fraction been 7918·2, or $\frac{79182}{10}$, the sum could not easily have been solved, for $\sqrt{10}$ is an irrational quantity. Similar reasoning gives us the following results:—

444. The square root of a number of two decimal places has one decimal place.

The square root of a number of four decimal places has two decimal places.

The square root of a number of six decimal places has three decimal places.

The square root of a number of eight decimal places has four decimal places.

Therefore if in any Evolution sum we want accuracy to a certain number of decimal places, we must have twice as many decimal places in the number whose root is required.

Whenever a number has an odd number of decimal places we may add a cipher—and by (253) this will not alter its value—and then, remembering the above rule, may neglect the decimal points, and proceed as in whole numbers.

445. *Example.*—Find the square root of 2710·382.

If we consider this whole number, without the decimal point, as the numerator of a fraction whose denominator is 1000, we shall not be able to express the answer decimally, for $\sqrt{1000}$ is not a suitable number for the denominator of a decimal fraction. But because $\sqrt{10000} = 100$, the root of 2710·3820, or of $\frac{27103820}{10000}$, will have 100 for its denominator.

$$
\begin{array}{rrl}
 & 27\acute{1}0\acute{3}8\acute{2}\acute{0} & (5000 + 200 + 6 = 5206) \\
5000^2 = & \underline{25000000} & \\
 & 2103820 & \\
10200 \times 200 = & \underline{2040000} & = (2 \times 5000 \times 200) + 200^2 \\
 & 63820 & \\
10406 \times 6 = & \underline{62436} & = 2 \times (5000 + 200) \times 6 + 6^2 \\
 & 1384 &
\end{array}
$$

5206 is the nearest root of 27103820, but as this latter number is the numerator of a fraction whose denominator is 10000, the denominator of its root is 100, and 52·06 is the nearest answer. It is evident that the answer might be carried to a greater degree of accuracy if other pairs of ciphers were added to the number, for every two places added to the number would add one to the root.

RULE FOR EXTRACTING THE SQUARE ROOT.

446. Place a point over the unit and over every second figure from it in both directions. If there be any decimal places, add a cipher if necessary, so that the last figure shall be pointed.

Find the nearest root of the first period, and subtract it. Add the next period to the remainder. Double the first part of the root, and find how many times it is contained in this new dividend, omitting the last figure. Add the quotient thus found to the divisor, multiply the divisor thus formed by the second figure of the quotient, subtract this product as before, and add the next period to the remainder. Proceed in this way until the last period has been brought down.

447. *Observation.*—In the examples just given we have set down more figures than are necessary in the working. The following example will show the ordinary or abridged process, and also the significance of each step of that process.

Example I.—Extract the root of 141376.

$$\begin{array}{rcrl}
 & & 1\acute{4}1\acute{3}7\acute{6} & (300 + 70 + 6) \\
3^2 & = & \underline{9} & \\
 & & 513 & \\
67 \times 7 & = & \underline{469} & = (2 \times 300 \times 70) + 70^2 \\
 & & 4476 & \\
746 \times 6 & = & \underline{4476} & = 2 \times (300 + 70) \times 6 + 6^2 \\
 & & \cdots &
\end{array}$$

Here, because 141376 has been found to contain

$300^2 + (2 \times 300 \times 70) + 70^2 + (2 \times \overline{300 + 70} \times 6) + 6^2$;

$\therefore 141376 = (300 + 70 + 6)^2 = 376^2$, nd $\sqrt{141376} = 376$.

Example II.—Find the square root of 7 to 3 places of decimals.

$$
\begin{array}{r}
7 \\
\underline{4} \\
300 \\
\underline{276} \\
2400 \\
\underline{2096} \\
30400 \\
\underline{26425} \\
3975
\end{array}
\qquad\qquad
\begin{array}{l}
\underline{|\ 2{\cdot}645\qquad\qquad} \\
\begin{array}{r|r|r}
46 & 524 & 5285 \\
6 & 4 & 5
\end{array}
\end{array}
$$

448. It is evident that there is no limit to this process, for, by the addition of pairs of ciphers, the root of any number which is not a perfect square, can be ascertained to as many places of decimals as may be desired.

449. In the case of decimals, adding twice as many figures as are needed in the root, is in effect multiplying the numerator of the fraction by the square of that number which represents the degree of accuracy required. When we say accuracy is required to 3 places of decimals, we in effect demand that it shall not differ so much as $\frac{1}{1000}$ from the truth; we therefore multiply both numerator and denominator by 1000000, which is the square of 1000, according to the principle enunciated in (441). In the example just given, because $\sqrt{7}$ has to be found true to $\frac{1}{1000}$, the sum takes this form:—

$$\sqrt{7} = \sqrt{\frac{7 \times 1000^2}{1000^2}} = \frac{\sqrt{7000000}}{1000} = \frac{2645}{1000} = 2{\cdot}645.$$

The following rules will help to show when a number has and when it has not a rational square root:—

450.—*No even number not divisible by* 4 *is a perfect square.*

For every even number may be represented by $2n$; its square therefore must always (420) be $4n^2$ and will always be divisible by 4.

451. *No odd number which, diminished by* 1, *is not divisible by* 4, *is a perfect square.*

Because every odd number may be expressed as $2n + 1$, the square of such a number must always be $4n^2 + 4n + 1$. This number diminished by unity is evidently divisible by 4.

452. *No number terminating in 2, 3, 7, or 8, is a perfect square.*

For if a square number does not terminate in a cipher, the unit figure in the square must have been obtained by squaring the unit figure of the root. And the squares of the nine digits end severally in 1, 4, 5, 6, or 9.

453. *A number ending in 5 cannot be a perfect square unless the* number in the tens place be 2.

For whenever 5 is the unit of a root, the square will always be 5^2, plus 2×5 multiplied into a certain number of tens; and 2×5, or 10, multiplied into tens will always give hundreds, and the square of such a number will consequently consist of $5^2 +$ a number of hundreds.

454. *No number terminated by an odd number of ciphers can be a perfect square.*

Because (444) a number ending with a cipher will always have twice as many ciphers at the end of its square.

455. *The square of every proper fraction must itself be a fraction, and cannot be a whole number.*

Because by (220) the product of two proper fractions is less than either of them.

456. *If the square of one number measure the square of another, the first number itself measures the second.*

For by (168) one number is measured by another when it contains all the prime factors of that other, and the square of a number contains no other prime factors than compose the number itself. Wherefore a^2 contains no other prime factors than are contained in a, and b^2 no others than are contained in b. Hence if a^2 measures b^2, b contains all the prime factors of a, and is therefore measured by it.

457. *If any fraction whatever be expressed in its lowest terms, its square cannot be a whole number.*

For if it could, then the square of $\frac{a}{b}$, or $\frac{a^2}{b^2}$, being a whole number, would have its numerator measured by its denominator; *i.e.*, b^2 would measure a^2 while b did not measure a, which would contradict the last proposition.

458. *If the square root of an integer number be not itself an integer, that root must be a surd or incommensurable quantity.*

For by (457) it cannot be a vulgar fraction, and therefore it cannot be a recurring decimal; because by (266) all recurring decimals can be exactly expressed as vulgar fractions.

459. *Observation.*—These results will be found true whatever be the scale of notation which we adopt. Numbers which are perfect squares on the decimal system would be so also on all others, whereas the roots of 3, of 7, of 11, are surds, or incommensurable numbers, on all scales alike.

Exercise CXXIII.

☞ Extract the square root of the following numbers:—

1. 17698849; 698485; 6084; 4906.
2. 841; 1287; 56821444; 714025.
3. 9585216; 45369; 10816; 745·29.
4. 3370·9636; 73008·05; ·000256; $\frac{2250}{3610}$.
5. 5·764801; 100996·84; ·047089.
6. 372344·04; 2768·412; 39705678.
7. 14590·2241; 9·8596; ·00591361.
8. 14·630625; 16·752649; ·006724.
9. $\frac{2}{5}$; $\frac{28}{29}$; $2\frac{1}{4}$; 15·8.
10. 6·4; ·64; ·064 ·0064; 640.
11. $12\frac{3}{5}$; $64\frac{1}{5}$; 18·27; 40·96.
12. $\frac{2·05}{2·25}$; $\frac{3·79}{·0016}$; $\frac{7·98}{52·4}$ $\frac{137}{219}$.
13. Find the square root of 11 true to $\frac{1}{15}$, and of 223 to $\frac{1}{40}$.
14. Find the square root of $79\frac{1}{8}$ true to $\frac{1}{20}$, and of $\frac{7}{13}$ to $\frac{1}{30}$.
15. Find the square root of 563 true to $\frac{1}{38}$, and of $413\frac{1}{9}$ to $\frac{1}{25}$.
16. Find $\sqrt{56}$ true to $\frac{1}{31}$, and $\sqrt{21\frac{3}{7}}$ true to $\frac{1}{11}$.
17. What is the difference between the square root of the sum of 1790 and 4451, and the sum of their square roots?
18. The product of two equal numbers is 509796. What are they?
19. Find the square root of 8 and of $17\frac{3}{8}$ to four places of decimals.

SECTION IV.—INVOLUTION TO THE THIRD POWER, OR FORMATION OF THE CUBES OF NUMBERS.

460. The cube or third power of a number is found by multiplying it by itself twice, or by finding the product of three equal factors.

Thus $4 \times 4 \times 4 = 64 = 4^3$ = the cube of 4.

461. The cubes of the digits (408) should be committed to memory, as they are needed throughout the Rule for the Extraction of the Cube Root.

EXERCISE CXXIV.

☞ Find by multiplication the cubes of the following numbers :—

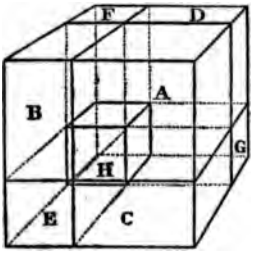
1. 17; 19; 30; 42; 16.
2. 18; 12; 3·7; 6·8; 4·15.
3. $2\frac{1}{4}$; $\frac{5}{8}$; $\frac{7}{12}$; $\frac{8}{15}$; $\frac{6}{11}$.

462. It is necessary first to consider a number consisting of *two* parts, and to find how its cube is made up of the cubes of those parts.

463. * *If a number be divided into two parts, the cube of that number equals the cube of the first, together with three times the square of the first multiplied by the second, together with three times the square of the second multiplied by the first, together with the cube of the second.*

---

* The analogy between number and geometry is again worth noting here. For let each of the equal edges of a cube be similarly divided into two parts, and let lines be drawn from the points of section; then, if the cube be cut into portions at these lines, it will be found that the whole has been divided into eight parts, corresponding to the products mentioned in the text. If we call the whole line 12 and the longer and shorter portions of the equal lines be respectively 8 and 4, it may be seen that the eight portions are as follow :—



A, cube of 8, or 8^3.

B, C, and D, pieces 8 long, 8 wide, and 4 thick $(3 \times 8^2 \times 4)$.

E, F, and G, pieces 8 long, 4 wide, and 4 thick $(3 \times 8 \times 4^2)$.

H, cube of 4, or 4^3.

Demonstrative Example.—Let 10 be divided into two parts, 6 and 4. Then $(6 + 4)^3$ may be found thus:—

$$\begin{array}{ll}
6 + 4 & \\
\underline{6 + 4} & \\
6_2 + 6 \times 4 & = (6 + 4) \times 6 \\
\underline{\quad 6 \times 4 + 4_2} & = (6 + 4) \times 4 \\
6^2 + 2\,(6 \times 4) + 4^2 & = (6 + 4) \times (6 + 4) = 10^2 \\
\underline{6 + 4} & \\
6^3 + 2\,(6^2 \times 4) + 6 \times 4^2 & = (6^2 + 4)^2 \times 6 \\
\underline{\quad 6^2 \times 4 + 2\,(6 \times 4^2) + 4^3} & = (6 + 4)^2 \times 4 \\
\left.\begin{array}{l} 6^3 + 3\,(6^2 \times 4) + 3\,(6 \times 4^2) + 4^3 \\ \text{or} \\ 216 \;+\; 432 \;+\; 288 \;+\; 64 \end{array}\right\} & = (6 + 4)^3 = 10^3 = 1000
\end{array}$$

General Formula.—Let $a = b + c$, then

$$a^3 = b^3 + 3b^2c + 3bc^2 + c^3.$$

EXERCISE CXXV.

☞ Divide the following numbers into two parts, and show how the cubes of the whole numbers are formed of the cubes of the parts:—

Example.—Because $5 = 3 + 2$,

$$\therefore 5^3 = 3^3 + (3 \times 3^2 \times 2) + (3 \times 3 \times 2^2) + 2^3.$$
$$125 = 27 + 54 + 36 + 8.$$

And because $26 = 20 + 6$,

$$\therefore 26^3 = 20^3 + (3 \times 20^2 \times 6) + (3 \times 20 \times 6^2) + 6^3.$$

1. 7; 9; 10; 12; 15.
2. 1·5; 63; 81; 24; 17.
3. 80; 21; 34; 13; 30.

464. *The difference between the cubes of any two numbers whose difference is* 1, *equals three times the square of the less, plus three times the less, plus one.*

For if in the former expression we substitute 1 for c,

$$\therefore (b + 1)^3 = b^3 + 3b^2 + 3b + 1.$$

Hence, for example, the difference between $(64)^3$ and $(65)^3$ equals $(3 \times 64)^2 + (3 \times 64) + 1 = 12481$.

465. *Observation.*—From this it may be seen how great a distance there is between the cubes of any two consecutive numbers, and how few numbers are perfect cubes.

EXERCISE CXXVI.

☞ (a). Verify the assertion in (464) in the case of the cubes of the first 10 numbers.

(b). Ascertain by this rule what is the difference between the cubes of the following pairs of numbers:—

1. 15 and 16; 21 and 22; 11 and 12; 17 and 18.
2. 30 and 31; 101 and 102; 24 and 25; 27 and 28.
3. 125 and 126; 19 and 20; 46 and 47; 100 and 101.

466. *The cube of a number cannot contain more than three times as many figures as the number itself, nor less than three times as many, minus two.*

Demonstrative Example.—It has been shown that the cube of 9, which is the highest number expressed by one digit, is 729, *i.e.*, it extends to three figures; while the cube of 10, the lowest number of two digits, is 1000, a number of the fourth place. In the same manner it may be seen that every number between 10 and 100 will have its cube between 1000 and 1000000, and that—

A number of one digit has in its cube not more than 3 digits.

A number of two digits has in its cube not more than 6 digits, nor less than 4.

A number of three digits has in its cube not more than 9 digits, nor less than 7.

A number of four digits has in its cube not more than 12 digits, nor less than 10.

General Formula.—If a number contains n digits, its cube will contain not more than $3n$ and not less than $3n - 2$.

467. And because $\cdot 1 = \frac{1}{10}$ and $(\frac{1}{10})_3 = \frac{1}{1000} = \cdot 001$
$\cdot 01 = \frac{1}{100}$ and $(\frac{1}{100})^3 = \frac{1}{1000000} = \cdot 000001$
$\cdot 001 = \frac{1}{1000}$ and $(\frac{1}{1000})^3 = \frac{1}{1000000000} = \cdot 000000001$

Therefore the cube of a fraction in the first decimal place extends to the third decimal place.

The cube of a fraction in the second decimal place extends to the sixth decimal place.

And the cube of a fraction in the third decimal place extends to the ninth decimal place.

468. *The cube of the product of two or more numbers is the same as the product of their cubes.*

Demonstrative Example.—Because $12 = 3 \times 4$,

therefore $12^3 = 3 \times 4 \times 3 \times 4 \times 3 \times 4$.

But (65) this product is the same with the factors in any order.

$\therefore 12^3 = 3 \times 3 \times 3 \times 4 \times 4 \times 4 = 3^3 \times 4^3$.

General Formula.—If $a = bcd$, then $a^3 = b^3c^3d^3$.

EXERCISE CXXVII.

☞ Resolve each of the following numbers into factors, and verify the assertion in (468):—

1. 6; 10; 12; 15; 16.
2. 14; 21; 24; 30; 18.
3. 25; 32; 56; 100; 108.

SECTION V.—EVOLUTION FROM THE THIRD POWER, OR EXTRACTION OF THE CUBE ROOT OF NUMBERS.

469. *If a number has for its cube root a number consisting of two parts, it must contain the cube of the first part; three times the square of the first part multiplied by the second part; three times the product of the first part and the square of the second, together with the cube of the second.*

This is only the same truth that is given in (463), adapted to the inverse process of Evolution. It will suffice to explain the reason of the rule in all cases, even though the cube root contain three or more parts. For because $10^3 = 1000$, the cube of any number of tens will give the cube of that number multiplied by 1000 (468), *e. g.*, the cube of 70 or of $7 \times 10 = 7^3 \times 10^3 = 7^3 \times 1000$. If, therefore, we cut off the last three figures of any integer number, and find the cube root of the rest, the answer multiplied by ten will give the cube root of that number of thousands. And because $100^3 = 1000000$, therefore the cube root of any number of millions equals the cube root of that number multiplied by 100. For by (468)

$$\sqrt[3]{27000000} = \sqrt[3]{27} \times \sqrt[3]{1000000} = \sqrt[3]{27} \times 100.$$

And if we cut off the last six figures of a number, and find the cube root of the rest, the answer will be a number which, multiplied by 100, will give the cube root of the millions.

Hence, to find the cube root of any number containing millions, we must first find the cube root of the millions and the thousands, and having obtained it, consider this root, though consisting of two figures as the first part, and then proceed to extract the root of the remaining parts

470. *If a point be made over the unit figure and over every third figure from it, the number of such points shows the number of digits in the cube root; e.g.,*

The cube root of $1\dot{8}52\dot{8}91\dot{6}54\dot{2}$ will contain four digits.

The cube root of $7\dot{0}96\dot{2}$ will contain two digits.

The cube root of $18\dot{7}$ will contain one digit.

471. The following examples will show how this principle is employed in the extraction of the cube root:—

Example I. —

$$\begin{array}{r|l}
 & 17\dot{5}61\dot{6}\ (50 + 6 \\
50^3 = & \underline{125000} \\
3 \times 50^2 = 7500) & \underline{\ 50616} \\
7500 \times 6 \ = & \ 45000 \\
3 \times 50 \times 6^2 = & \ \ \ 5400 \\
6^3 = & \ \ \ \ \ \underline{216} \\
 & \ 50616
\end{array}$$

On placing points it is observed that the answer will contain two figures, tens and units. We first find the nearest cube root of the first period, 175000. This gives 5, which is therefore 5 tens. On subtracting the cube of 50, 50616 is found to remain. But by (469) this number must contain three times the square of 50 multiplied by the new part, plus three times 50 multiplied by the square of the new part, plus the cube of the new part. In order to find, approximately, what this new part is, we divide 50616 by three times the square of 50. When the quotient, 6, has thus been obtained, it becomes necessary to subtract three several sums from the 50616, viz., $3 \times 50^2 \times 6$, which is 45000; then $3 \times 50 \times 6^2$, which is 5400; and lastly, 6^3, which is 216. But the sum of these numbers equals the former remainder.

By (464) it has been shown that if $a = (b + c)$,
Then $a^3 = b^3 + 3b^2c + 3bc^3 + c^3$.

And the number 175616 has been shown to contain—

$$50^3 + (3 \times 50^2 \times 6) + (3 \times 50 \times 6^2) + 6^3.$$

Therefore $175616 = (50 + 6)^3$, and $56 = \sqrt[3]{175616}$.

472. *Example II.*—Find the cube root of 33698267.

$$\begin{array}{rrl}
 & & 3\dot{3}69\dot{8}26\dot{7}\ (323 \\
 & 3^3 = & 27 \\
3 \times 30^2 = & 2700 & \overline{6698} \\
3 \times 30 \times 2 = & 180 & \\
2^2 = & \underline{4} & \\
 & 2884 \times 2 = & \underline{5768} \\
 & & 930\ 267 \\
3 \times 320^2 = & 307200 & \\
3 \times 320 \times 3 = & 2880 & \\
3^2 = & \underline{9} & \\
 & 310089 \times 3 = & \underline{930\ 267} \\
 & & \ldots\ \ \ldots
\end{array}$$

The answer here will contain three digits. The nearest cube root of the millions is found to be 3 (or 300). On subtracting 27, and bringing down the remaining thousands, we have 6698. These are thousands, and the cube root of thousands being tens, we have next to find the number of tens in the root; let this number be x.

Then (469) 6698 ought to contain $(30^2 \times 3x) + (30 \times 3x^2) + x^3$. But x being a common factor of these quantities, it will be convenient to consider it as $(\overline{30^2 \times 3} + \overline{30 \times 3x} + x^2)\,x$, as by this means we avoid three several multiplications by the same number. To find the required number we try $30^2 \times 3$, or 2700 into 6698; the nearest quotient is 2. Place the 2 as the answer in the tens place, as x. Then multiply it by the sum of 3×30^2, $3 \times 30 \times 2$, and 2^2. Subtract this number and bring down the next period.

It has now been found that 33698267 contains the cube of 32 tens, or 320, and also 930267 besides. 320 may now be considered as the first part of the root, and we have to find the other part. This will be a unit figure, for it is evident that the number 3 would have been too great in the tens place, and therefore that the answer is less than 330 and more than 320. Let the unknown unit be y; then by (470) 930267 ought to contain $3 \times 320^2y + (3 \times 320y^2) + y^3$, or as y is a common factor $(\overline{3 \times 320^2} + \overline{3 \times 320y} + y^2)\,y$. In order to find y we take as a trial divisor 3×320^2, which is 307200, and find how many times it is contained in 930267; the quotient seems to be 3. On taking from the last remainder these three several portions we find no remainder; 323 is therefore the exact root required.

For because generally (464) the cube of $x + y + z =$
$(x + y)^3 + 3(x + y)^2 z + 3(x + y) z^2 + z^3$;

And $(x + y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3$;

Therefore $(x + y + z)^3 =$
$x^3 + 3x^2 y + 3xy^2 + y^3 + 3(x + y)^2 z + 3(x + y) z^2 + z^3$;

And $(300 + 20 + 3)^3 = 300^3 + (3 \times 300^2 \times 20) + (3 \times 300 \times 20^2) + 20^3 + (3 \times 320^2 \times 3) + (3 \times 320 \times 3^2) + 3^3$.

Hence, as 33698267 has been found to contain all these several parts—

$$\sqrt[3]{33698267} = 300 + 20 + 3 = 323.$$

473. *The cube root of a fraction may be found by taking the cube roots of its numerator and denominator respectively.*

Demonstrative Example.—Because $\left(\frac{3}{5}\right)^3 = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$;

$$\therefore \sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{3}{5}.$$

General Formula.— $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$.

474. As in the Extraction of the Square Root, it is usual to reduce all vulgar fractions whose cube root is required into a decimal form, and to let them be so expressed that their denominators shall have a rational cube root.

From (466) it may be seen that the only powers of 10 which have a rational cube root are the third, the sixth, the ninth, the twelfth, &c., and that the cube root of a number extending to the third decimal place has itself one decimal place; that of a number extending to the sixth has two places, &c. Hence, before extracting the root of a decimal fraction, it is necessary to place three times as many ciphers as there are figures required in the root, and that the number of decimal places should always be a multiple of 3.

475. *Extraction of the Cube Root by Approximation.*—The method of obtaining the cube root of a number true to any fraction required (449) is as applicable here as for the square root. For to find the cube root of any number true as far as any given fraction, say $\frac{1^3}{n}$ we may multiply

the number by the cube of *n*, extract its root, and then divide this answer by *n*. If, for example, it is required to find the cube root of 15 true to $\frac{1}{12}$, the number 15 may take the form $\frac{15 \times 12^3}{12^3}$ or $\frac{25920.}{12^3}$ Now as the nearest cube root of 25920 is 29, the root required is $\frac{29}{12}$, or $2\frac{5}{12}$.

Example II.—To find the cube root of $37\frac{8}{13}$ true to $\frac{1}{20}$.

Multiply $37\frac{8}{13}$ by 20^3.

$$\tfrac{489}{13} \times 20^3 = 489 \times \tfrac{8000}{13} = \tfrac{3912000}{13} = 300923\tfrac{1}{13}.$$

But $\sqrt[3]{300923} = 67$. Hence $\sqrt[3]{\frac{489}{13}} = \frac{67}{20}$, or $3\frac{7}{20}$ nearly.

Example III.—In like manner the cube root of 47 true to $\frac{1}{20}$ =

$$\sqrt[3]{\frac{47 \times 20^3}{20^3}} = \tfrac{72}{20} = 3\tfrac{3}{5}.$$

Example IV.—The cube root of $23\frac{7}{8}$ true to $\frac{1}{13}$ =

$$\sqrt[3]{\frac{191 \times 13^3}{8 \times 13^3}} = \tfrac{37}{13} = 2\tfrac{11}{13}.$$

Example V.—The cube root of $\frac{5}{7}$ true to $\frac{1}{30}$ =

$$\sqrt[3]{\frac{5 \times 30^3}{7}} \div 30 = \tfrac{9}{10}.$$

Questions are seldom given in this form, but the common method of treating all fractions as decimals is evidently founded on the same principle. Thus if we are required to find $\sqrt[3]{25}$ true to two places of decimals, this means that the answer is not to differ from the truth by so much as $\frac{1}{100}$, we therefore multiply 25 by the cube of 100, when it becomes 25000000; the nearest root of this is 292; on dividing this by 100 we have 2·92 as the required answer. The process in principle is identical with the preceding.

For $\sqrt[3]{25}$ true to $\frac{1}{100}$ = $\sqrt[3]{\frac{25 \times 100^3}{100^3}} = \sqrt[3]{25 \times 100^3} \div 100 =$ $292 \div 100 = 2{\cdot}92$.

In like manner $\sqrt[3]{3{\cdot}1415}$ true to $\frac{1}{100}$ = $\sqrt[3]{\frac{3{\cdot}1415 \times 100}{100^3}}$

$$\sqrt[3]{3141500} \div 100 = 1{\cdot}46.$$

TO EXTRACT THE CUBE ROOT OF A NUMBER—

RULE I.

476. Place a point over the unit and over every third figure from it both ways. The number of points shows the number of figures in the root.

Find the nearest cube root of the first period, set it down as the first figure of the root, subtract its cube from the first period, and bring down the next period to the right of the remainder.

Take three times the square of the first found figure, considered as a number of tens, and ascertain how many times it is contained in this new dividend. The quotient will form the second figure of the root.

Take three times the square of the first part + three times the product of the first and the second + the square of the second. Multiply this sum by the second figure of the root, and subtract the result.

If more periods remain to be brought down, repeat the same process, considering that part of the root already found as a number of tens.

Example.—Extract the cube root of 7 to three decimal places.

7 (1·912
1
0000

$3 \times 10^2 = 300$
$3 \times 10 \times 9 = 270$
$9^2 = 81$
$651 \times 9 = 5859$
141000

$3 \times 190^2 = 108300$
$3 \times 190 = 570$
$1^2 = 1$
$108871 \times 1 = 108871$
32129000

$3 \times 1910^2 = 10944300$
$3 \times 1910 \times 2 = 11460$
$2^2 = 4$
$10955764 \times 2 = 21911528$
10217472

Answer 1·912.

RULE II.

477. Proceed as in the former rule, to find the first part of the root, and to find the second part by trial.

To three times the first found figure annex the second; multiply by the second, add this product to the trial divisor, and multiply the whole sum by the second figure of the root.

To find the next trial divisor, add together the former complete divisor, the sum that completed it, and the square of the second part.

Having thus found the third part of the root, add it to the right hand of three times the former parts; multiply this sum by the third part, add the result to the last trial divisor, and multiply by the third part. Proceed in the same manner with other parts.

Observation.—This is an abridged form of the same process as the last. The reason will be easily seen in the following—

Example.—Find the cube root of 48228550.

$$\begin{array}{lrrl}
 & & & 48228550\ (364 \\
 & & & \underline{27} \\
\text{1st Trial divisor} & 3 \times 30^2 = & 2700 & 21228 \\
 & (a)\ (3 \times 30 \times 6) + 6^2 = & \underline{576} & \\
 & & 3276 \times 6 = & \underline{19656} \\
 & & \underline{36} & 1572550 \\
\text{2nd Trial divisor } (b) & 3 \times 360^2 = & 388800 & \\
 & (c)\ (3 \times 360 \times 4) + 4^2 = & \underline{4336} & \\
 & & 393136 \times 4 = & \underline{1572544} \\
 & & & 6
\end{array}$$

$$\begin{array}{lr}
(a) & 30 \\
 & \underline{3} \\
(3 \times 30) + 6 = & 96 \\
 & \underline{6} \\
(3 \times 30 \times 6) + 6^2 = & 576
\end{array}$$

(*b*) This number, 388800, is clearly 3 times the square of 360, for it contains 3 times the square of 300 (270000), 6 times the product of 300 and of 60 (for 57600 is added in twice), together with three times the square of 60.

$$\begin{array}{lr}
(c) & 360 \\
 & \underline{3} \\
(3 \times 360) + 4 = & 1084 \\
 & \underline{4} \\
(3 \times 360 \times 4) + 4^2 = & 4336
\end{array}$$

EXERCISE CXXVIII.

☞ Find the cube roots of the following numbers :—

1. 32768 ; 1157·625 ; 247673152.
2. 32461759 ; 78610563187 ; 5·088448.
3. 1·29503 ; 4173·281 ; 135796·744.
4. 8108486729 ; 165795999168.
5. 1·25 ; 3 ; 2197583 ; ·4.
6. 2599609375 ; 247791486041.
7. 356·702522688 ; 219038133·952.
8. 138·348848448 ; 51645·087424.
9. Find the cube root of $\frac{5}{7}$ true to $\frac{1}{16}$, and that of $31\frac{2}{5}$ true to $\frac{1}{25}$.
10. Find $\sqrt[3]{\frac{4}{9}}$ true to $\frac{1}{30}$, and $\sqrt[3]{\frac{11}{15}}$ true to $\frac{1}{45}$.
11. Find $\sqrt[3]{24\frac{1}{7}}$ true to $\frac{1}{15}$, and $\sqrt{273\frac{1}{4}}$ to $\frac{1}{32}$.
12. Find the approximate cube root of the following numbers in a decimal form :—
    (*a*). 729863 ; 527410 ; 6283·0542.
    (*b*). $\frac{3}{5}$; ·68 ; ·0654 ; 27·0298.
    (*c*). 17·$\dot{4}$; 18·$\dot{0}3\dot{6}$; 509·$\dot{7}$; 62 87$\dot{5}$.
13. The product of 3 equal numbers is 3189506048, what are they ?

Questions on Involution and Evolution.

Define the terms power, root, involution, evolution, square, cube, exponent. Give a reason why the product of two equal numbers should be called a square, and that of three such numbers a cube. When are quantities rational? when irrational? Why are these terms applied? What is a surd?

On what fundamental axiom of arithmetic does Involution rest? Enunciate in order the propositions relating to the square of a number consisting of two parts. Give examples. Suppose a number consist of 1, of 4, of 12 parts, describe how its square is formed. By how much does the square of a number differ from the square of the number next above it? How is the square of a number related to the squares of its factors?

How many digits will be contained in the square of a number containing two digits? Why? What use is made of this fact in extracting the square root? What is the product of the sum and difference of two numbers? What is the square of their difference? By how much do the square of the sum, the sum of the squares, and the square of the difference of two numbers differ from one another? How is the square of half a number related to that of the number itself? How is the square of half the sum of two numbers related to their product?

How is the root of a fraction to be found? What is the most convenient form in which to extract such a root? Explain the reason of each of the following portions of the ordinary rule—pointing, making an even number of decimal places, doubling the part of the root, adding a new part, and multiplying by the new part. How may a root be obtained approximately? Suppose accuracy be required as far as $\frac{1}{n}$, how may it be obtained? By what tests may you ascertain, by inspection, whether a number is a perfect square or not?

Describe the manner in which the third power of a number is made up. How do the cubes of any two consecutive numbers differ? In the extraction of the cube root, why should we point every third figure? Give a reason for adding ciphers in the case of decimal fractions, and for tripling the square of the first found part. Take the number 178614, extract its square and also its cube root, analyzing the process in each sum, so as to show the separate meaning of every line.

Express in words the truths embodied in the following formulæ :—

1 $(a + b)^2 = (a + b)\,a + (a + b)\,b = a^2 + 2a\,b + b^2.$

2. $(a + b + c + d + e)^2 = a^2 + 2a\,(b + c + d + e) + b^2 + 2b\,(c + d + e) + c^2 + 2c\,(d + e) + d^2 + 2de + e^2.$

3. $(a + b)\,(a - b) = a^2 - b^2$, and $(a - b)^2 = a^2 - 2ab + b^2.$

4. $\left(\frac{a + b}{2}\right)^2 - ab = \left(\frac{a - b}{2}\right)$, $(a + 1)^2 - a^2 = 2a + 1.$

5. $(abc)^2 = a^2b^2c^2$, $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, and $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$

6. $(a + b)^3 = a^3 + 3a^2b + {}^3ab^3 + b^3.$

7. $abcd^3 = a^3b^3c^3a^3$, and $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}.$

APPLICATION OF ARITHMETIC TO GEOMETRICAL MEASUREMENTS.

478. Many of the preceding rules are employed in the measurement of lines, surfaces, and solids.

Geometry investigates the proportions subsisting among the lines by which regular plane and solid figures are contained, and shows how those lines determine the magnitude of the figures themselves.

479. Every practical rule of mensuration is based upon some theorem of geometry, and arithmetic can only be applied to this purpose when we take for granted the truth of some such theorem. The following are among the most important of these, with their practical applications :—

480. *I. Every parallelogram, whatever be its form, is equal in area to a rectangle, having a base and perpendicular altitude equal to its own.* (Euclid, i. 36.)

The area of every triangle is half of that of a parallelogram having the same base and altitude as itself. (Euclid, i. 41.)

Parallelograms are to one another in a ratio compounded of their bases and altitudes. (Euclid, Book vi. 1, *corollary.*)

481. Hence, TO FIND THE AREA OF A PARALLELOGRAM—

Multiply the number of linear units in the base by the number of linear units in the altitude.

And, TO FIND THE AREA OF A TRIANGLE.

Find the area of a parallelogram having the same base and altitude, and divide the result by 2.

482. The foot is generally taken as the standard of linear measure by artisans, and the square foot is the principal superficial unit. Each of these units is subdivided into twelfths, twelfths of twelfths, &c. Hence the ordinary method of making such calculations is called *duodecimal.*[*] The two subdivisions are as follows :—

---

[*] From Latin *duodecim,* twelve, or 2 (*duo*) + 10 (*decem*).

1 foot = 12 *primes* or inches = 144 seconds = 1728 thirds, &c.
1 sq. foot = 12 primes = 144 *seconds* or inches = 1728 thirds, &c.

The *prime*, or twelfth of a *linear* foot, is a *linear* inch, but the *second*, or 144th of a *square* foot, is a square or superficial inch, the prime or twelfth of a square foot containing 12 square inches.

483. It is usual to express these subdivisions by accents, thus :—

$$7 \text{ feet } 6'\ 3''\ 2'''\ 8'''' = 7 + \frac{6}{12} + \frac{3}{12^2} + \frac{2}{12^3} + \frac{8}{12^4} \text{ feet.}$$

Now because a rectangular area 1 foot long and 1 foot wide is a square foot—

∴ *Feet multiplied by feet give square feet.*

And because such an area 1 foot long and 1 inch wide contains 12 square inches—

∴ *Feet multiplied by inches give superficial primes.*

And because such an area 1 inch long and 1 wide is a square inch—

∴ *Inches multiplied by inches give superficial seconds.*

And because such an area 1 inch long and 1 second wide contains one-twelfth of a square inch—

∴ *Inches multiplied by seconds give superficial thirds.*

And in the same manner it may be seen that—

Feet multiplied by seconds give superficial seconds or square inches.

$$\left.\begin{array}{l} \textit{Seconds multiplied by seconds, } \text{or} \\ \textit{Inches multiplied by thirds} \ \ldots\ldots \end{array}\right\} \textit{give superficial fourths.}$$

Example I.—Find the area of a floor 17 feet 2′ 6″ by 11 feet 8′ 9.″

$$\begin{array}{rrrrrl}
\text{ft.} & \text{in.} & \text{sec.} & & & \\
17 & 2 & 6 & & & \\
11 & 8 & 9 & & & \\
\cline{1-3}
189 & 3 & 6 & & & = 189 + \frac{3}{12} + \frac{6}{144} = (17\ 2'\ 6'') \times 11 \\
11 & 5 & 8 & & & = 11 + \frac{5}{12} + \frac{8}{144} = (17\ 2'\ 6'') \times 8' \\
1 & 0 & 10 & 10''' & 6'''' & = 1 + \frac{10}{144} + \frac{10}{1728} + \frac{6}{20736} = (17\ 2'\ 6'') \times 9'' \\
\cline{1-5}
201 & 10' & 0'' & 10''' & 6'''' &
\end{array}$$

Example II.—What surface is contained by a triangle whose base is 18 feet 9 inches, and perpendicular height 14 feet 7 inches?

$$\begin{array}{rrrrrl}
 & \text{ft.} & \text{in.} & & & \\
 & 18 & 9 & & & \\
 & 14 & 7 & & & \\
\cline{2-3}
 & 262 & 6' & & & = (18\ 9') \times 14 \\
 & 10 & 11' & 3'' & & = (18\ 9') \times 7' \qquad \text{[base and altitude.} \\
\cline{1-4}
2) & 273 & 5' & 3'' & & = \text{area of rectangle having the same} \\
\cline{2-4}
\text{Square feet,} & 136 & 8' & 7'' & 6''' & = \text{area of triangle.}
\end{array}$$

484. When the fractions of a foot, given in the question, can be easily expressed in decimals, the area can be found more readily than by this method.

Example III.—Find the area of a surface measuring 7 feet 6 inches by 11 feet 9 inches.

Duodecimal method.

ft.	in.		
7	6		
11	9		
82	6		$= (7\ 6') \times 11$
5	7	6″	$= (7\ 6) \times 9'$
88	1′	·6″	$= 88\frac{18}{144}$ feet.

Decimal method.

$$\begin{array}{r} 11{\cdot}75 \\ 7{\cdot}5 \\ \hline 5875 \\ 8225 \\ \hline 88{\cdot}125 \end{array} = 88\tfrac{125}{1000} \text{ feet.}$$

485. As cost of paving, flooring, or tiling is usually calculated at a certain rate per yard or foot, it is generally convenient, when the question refers to price, to reduce the whole of the dimensions to one denomination, and represent them all as fractions of a foot.

Example IV.—What would be the cost of carpeting a room 32 feet 6 inches long and 23 feet 9 inches wide, at 5s. 6d. per square yard?

I. By Ordinary Fractions.

$32\frac{1}{2} \times 23\frac{3}{4}$ feet
$= (\frac{65}{2} \times \frac{95}{4})$ square feet
$= 771\frac{7}{8} \times \frac{1}{9} =$ square yards
$= 85\frac{55}{72}$ square yards

$(85\frac{55}{72} \times 5\frac{1}{2})$ s. = cost of carpeting
= £23 11s. $8\frac{1}{4}$d.

II. Duodecimally.

	ft.	in.	sec.
	32	6	
	23	9	
	747	6	
	24	4	6
9)	771	10	6
Square yards	85	9	3

		£	s.	d.
85 at 5s. 6d.	=	£23	7	6
$\frac{9}{12}$ …………	=	0	4	$1\frac{1}{2}$
$\frac{3}{144}$ …………	=	0	0	$0\frac{3}{4}$
		£23	11	$8\frac{1}{4}$

EXERCISE CXXIX.

1. Find the area of a rectangle 2 ft. 6′ 8″ by 17 ft. 8′ 10″, and of one 3 ft. 4′ 7″ by 1 ft. 9′ 11″.

2. Find the area of a rectangle 36 ft. 7′ 11″ by 9 ft. 8′ 3″, and of one 20 ft. 8′ by 15 ft. 0′ 10″.

3. Find the area of a rectangle 111 ft. 1′ 5″ by 14 ft. 7′, and of one 10 ft. 8′ 7″ 6‴ by 35 ft. 8′ 10″.

4. Find the area of a rectangle 7 ft. 6′ 9″ by 13 ft. 9′ 2′ 7″, and of one 20·72 feet by 18·71 inches.

5. Find the area of a rectangle 17·25 feet by 6·231 yards, and of one 238·19 feet, by 61·518 feet.

6. Find the area of a rectangle $72\frac{1}{4}$ feet by $56\frac{3}{4}$ feet, and of one $86\frac{1}{12}$ feet by $75\frac{3}{8}$ feet.

7. What is the area of a triangle whose base is 14·672 feet and perpendicular height 11 98 feet?

8. What is the area of a triangle whose base is 3·08 feet and perpendicular height 6·589 feet?

9. What is the area of a triangle whose base is 5·27 yards and perpendicular height 3·896?

10. What is the area of a triangle whose base is 37 ft. 6′ 11″ and perpendicular height 17 ft. 8′ 3″?

11. What is the length of a room containing 47 square yards of flooring if its breadth be 18 ft. 5′?

12. How wide is a rectangle containing 156 ft 8′ if its length be 17 ft. 9′ 4″?

13. How wide is a room containing 392 square ft. 5′ 8″ if its length be 23 ft. 8 in.?

14. An area is $250\frac{5}{8}$ feet, its length $17\frac{1}{2}$ feet; find the breadth.

15. What length of carpet $\frac{3}{4}$ of a yard wide will cover a floor $6\frac{1}{2}$ yards long by $5\frac{1}{4}$ wide?

16. 69 yards of carpet $\frac{3}{4}$ of a yard wide cover a room $10\frac{1}{2}$ yards long; how wide is the room?

17. What is the area of a passage whose length is 12 feet 6 in. and breadth 2 ft. 9 inches, and what would it cost to cover it at 5s. 6d. per square yard?

18. What will be the expense of a floor whose length is $10\frac{3}{8}$ yards and breadth $5\frac{1}{3}$, the price of paving being 2s. per square foot?

19. How many square feet of paper will cover the walls of a room 20 feet 10 inches long, 16 feet broad, and 10 feet 8 inches high?

20. What will remain out of 393 square feet of carpeting after covering a floor 23 feet 8 inches by 16 feet 7 inches?

21. Find the cost of covering a room $23\frac{3}{4}$ feet long by $16\frac{1}{4}$ wide, at 2s 9d. per square yard.

22. What is the cost of carpeting a room $16\frac{1}{2}$ feet square, at 4s. $10\frac{1}{2}$d. per yard, if the carpeting be 2 feet wide?

23. What length of carpet $\frac{3}{4}$ yard wide will cover a room 42 feet 5 inches long by $31\frac{3}{4}$ wide?

24. Compare the magnitudes of two triangles, the one having a base 17½ and an altitude 13¾, and the other having a base 20⅛ and an altitude 11⅝.

25. Compare the magnitude of a parallelogram 25 ft. 6′ by 17 ft. 7′ with that of a square on a line 23·67 feet long.

26. Compare the magnitudes of two rectangles, one of which is 17 ft. 8′ 7″ by 23 ft. 5′, and the other 11 ft. 6′ by 27 ft. 7′ 11″.

27. Find the difference between 27 square feet and 27 feet square.

28. Find the price of a rectangular piece of ground 52 feet 8 inches by 34 feet 9 inches, at 25s. per square foot.

29. How broad is a yard 36 feet 6 inches long, when the cost of paving it is £12 6s. 6¼d., at 5s. 3d. per square yard?

30. Find the cost of paving a yard 3 feet 10 inches broad and 12 feet 11 inches long, at 1s. 1d. per square foot.

486. *II. The square described on the greatest side of a right-angled triangle is equal to the sum of the squares described upon the other two.* (Euclid, i. 47.)

TO FIND ANY SIDE OF A RIGHT-ANGLED TRIANGLE—

487. The number representing the linear units in the hypothenuse equals the square root of the sum of the second powers of the number of linear units in the other two sides.

EXERCISE CXXX.

1. Find the hypothenuse of a right-angled triangle whose sides are 15 and 7, and of another whose sides are 9 and 16.

2. The two sides of a right-angled triangle are 120·5 and 83: by how much does the hypothenuse exceed the greater of those sides?

3. If the hypothenuse be 26, and one side 9, what is the other? If the hypothenuse be 153, and the base 64, what is the altitude?

4. The foot of a ladder 30 feet long is 14 feet from a house, and its top reaches the upper part of a circular window; when it is drawn away to a distance of 17 feet, the top reaches the lower edge of the window: what is the diameter of the window?

5. How long should a rod be which joins the extremities of two walls, of which one is 8 feet and the other 6 feet, if the two walls meet at a right angle?

6. Find the diagonals of 3 squares whose sides respectively are 14, 20·6, and 33 feet in length.

7. What are the sides of three squares whose diagonals are respectively 25, 57·6, and 85·4 feet in length?

8. By how much does the square described on a line 5·73 inches long exceed one on a line half of the length?

9. What is the length of the side of a square piece of ground equal in area to a rectangle containing 4970·25 square yards?

10. If a rectangle contains 14723 square yards, and one side measures 506 yards, what is the length of the diagonal?

11. By how much does the square described upon the diagonal of a square exceed the square itself?

12. A rectangle has for one side a line 25 feet, and for another a line 9·9 yards long. Compare the magnitudes of the squares described on the two sides, and the diagonals, respectively.

13. If two vessels sail from the same port, the one 17½ leagues due east, and the other 41·6 leagues due north, how far will they be apart?

14. If a ladder 80 feet long be so placed as to reach a window 40 feet high on one side of the street, and 30 feet high on the other, how wide is the street?

15. What are the sides of two squares whose areas are 5083·6 inches and 17·8 feet respectively?

16. The sum of the diagonals of two squares is 52 inches, and the difference 7 inches; compare the lengths of their sides respectively.

488. *III. The ratio of the diameter to the circumference of every circle is nearly that of* 1 *to* 3·14159.[*]

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[*] This may be proved in two ways:—(1.) As a deduction from the formulæ in analytical trigonometry, which give a numerical equivalent in terms of the radius, for the sine of an angle so small that the arc does not appreciably differ from the sine itself: and (2.) By actual measurement. On unrolling the circumference of a circle, or finding a straight line of the same length, it is found impossible to express *precisely* the relation between this line and the diameter. But if the diameter contain 10 parts, the circumference is observed to contain rather more than 31 such parts; if the diameter be 100, the circumference rather exceeds 314; if the diameter be 1000 the circumference is 3141; if the diameter has 100000 parts the circumference has 314159, &c. Each is a nearer approximation to the truth than its predecessor, but the two lines are incommensurable, however far the comparison be carried. The following number expresses the truth as far as to 35 decimal places:—

3·14159265358979323846264338327950288.

489. TO FIND EITHER THE CIRCUMFERENCE OR DIAMETER OF A CIRCLE WHEN THE OTHER IS KNOWN—

Multiply the diameter by 3·14159 to find the circumference, or divide the circumference by 3·14159 to find the diameter.

EXERCISE CXXXI.

1. Find the circumferences of circles whose diameters measure 17, 53·6, 247, and 10·9 feet respectively.

2. Find the diameters of circles whose circumferences measure 154, 208·6, and 4058 feet respectively.

3. The length of $\frac{1}{360}$ of the earth's circumference is about $69\frac{1}{12}$ miles; what is the earth's diameter?

4. What is the diameter of a circle whose semi-circumference is 54 yards?

5. Find the length of the French unit of length (*a metre*), which is a ten-millionth part of the distance from the equator to the pole, on the supposition that the diameter of the earth is 7,912 miles.

6. How many degrees are there in an arc whose length is 73·5, the radius of the circle measuring 115·6?

7. Find the radius of a circle whose circumference is equal to one-third of a mile.

8. Find the length of the quadrant of a circle, one-eighth of whose radius is 9·75 yards?

9. What is the length of an arc which measures $\frac{1}{8}$ the circumference, when $\frac{3}{4}$ of the radius is 70·25 feet?

10. If the semi-circumference of a circle, whose radius is 98·5 feet, is equal in length to the quadrant of another, what is the diameter of this last?

11. What is the length of the arc of a circle whose radius is 24 ft. 6 in., and which contains the same number of degrees as the arc of another circle whose length is 5 ft. 4 in., to radius 9·12 ft.?

12. What is the number of degrees in the arc of a circle whose diameter is 79 ft. 10 in., and which is equal in length to the arc of another circle containing $26° 4' 13''$, to a diameter of 98 ft. 8 in.?

13. Find the length of $17° 30'$ on the equator, on the supposition that a degree measures 69·15 miles.

490. *IV. The areas of circles are to one another as the squares of their diameters.*

*The area of every circle is equal to that of a rectangle whose base is the radius, and whose altitude is equal to one-half of the length of the circumference; or it is equal to that of a rectangle whose base is the diameter, and whose altitude is one-fourth of the circumference.**

491. TO FIND THE AREA OF A CIRCLE—

Multiply the square of the radius by 3·14159, or multiply the square of the diameter by ·7854.

Note.—·7854 = 3·14159 ÷ 4.

EXERCISE CXXXII.

1. Find the areas of three circles whose radii are 7, 8, and 9 respectively.

2. Find the area of three circles whose diameters are 75, 80·5, and 7·19 respectively.

3. If the radius of a circle be 3·5 feet, find the side of a square which shall have the same area.

4. What is the diameter of a circular field containing 2½ acres?

5. Compare the magnitudes of three circular plots of ground whose radii are 17, 18, and 19 respectively.

6. The diameters of two concentric circles are 173 and 191·6 feet; find the space included between the two circumferences.

7. What is the area of a gravel walk round a circular grass plot whose radius is 23 ft., the width of the walk being 5 ft. 6 in.?

8. What is the area of a path round a circular flower-bed whose diameter is 11·72 feet, the width of the path being 3 ft. 8 in.?

9. The area of the space occupied by a circular tower is 687 yards, the area of a moat round it is 1280 yards; what is the width of the moat in feet?

10. A man wishing to find the number of acres covered by a circular pond, walked round it at the rate of 3½ miles an hour, and found that it took him 2½ hours to complete the journey: required the area of the pond.

---

* These propositions are deducible from Euclid, Book vi., prop. 20.

11. How many acres are there in a circular field, the cost of planting a hedge round it being £72 10s. at 2s. 4d. per yard?

12. I have purchased a circular field for £690 12s. 6d., at £65 per acre; what will be the cost of digging a ditch round it at 5d. per yard, and of dividing it into two equal parts by a wall which will cost 4s. 9d. per yard in building?

13. Twelve persons can sit round a circular table, allowing 22 inches for each; what will be the price of a cloth to cover it, and extend 1 foot over the edge all round, at 2s. 6d. per square yard?

492. *V. The product of the numbers representing the three dimensions of a rectangular solid, or parallelopiped, represents the number of cubic units in the solid itself.**

493. TO FIND THE CUBIC CONTENTS OF A PARALLELOPIPED—

Find the continued product of the numbers representing the length, breadth, and thickness of the solid.

494. It is usual to employ the duodecimal method described in (482) to express the solid contents; a cubic foot being divided into *primes*, *seconds*, *thirds*, &c. Only here it must be remembered that whereas a lineal prime, or $\frac{1}{12}$ of a foot, is a lineal inch; a superficial second, or $\frac{1}{144}$, is a superficial inch; so a cubical third, or $\frac{1}{1728}$, is a cubic inch.

Example.—How many cubic feet and inches are there in a solid whose breadth is 9 ft. 3 in., length 11 ft. 5 in., and height 3 ft. 2 in?

	ft.	in.			
	9	3			
	11	5			
	101	9			
	3	10	3		
Square feet	105	7′	3″		= Superficial content of one side of the solid.
	3	2			
	316	9	9		
	17	7	2	6	
Cubic feet	334	4′	11″	6‴	= Solid content of the whole figure.
		12			
Reducing the primes and seconds to thirds or cubical inches.		59″			
		12			
		714‴			Ans. 334 cubic ft. 714 cubic in.

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* See note *ante*, p. 240.

EXERCISE CXXXIII.

1. What are the solid contents of a rectangular mass 15 ft. 6 in. long, 18 ft. 5 in. wide, and 23·5 ft. thick?

2. Find the content of a block of stone 17 ft. 9 in. long, 14 ft. 3 in. broad, and 5 ft. 6 in. thick; and its price at 4d. per cubic foot.

3. The weight of the cubic foot of water is about 1,000 ounces; what weight of water will fill a cistern 4 ft. 6 in. long, 3 feet broad, and 4 ft. 3 in. deep?

4. What will be the cost of a marble block 37 ft. 8 in. long, 8 feet broad, and 6 ft. 5 in. thick, at 5s. 6d. per solid foot?

5. If a brick be 9 in. long, 4 in. wide, and 3 in. thick, how many will be required for a wall 1 foot 10 in. thick, 100 yds. long, and 4½ yards high?

6. The weight of a cubic foot of Portland stone is 156 lbs.; find the weight of a block 7 ft. long, 3 ft. 9 in. broad, and 2 ft. 1 in. thick.

7. The weight of a cubic foot of oak is 58 lbs.; what is the weight of 3 beams, each being 12 ft. 6 in. long, 2 ft. 3 in. broad, and 1 ft. 6 in. thick?

8. Gold sells at £3 17s. 6d. per oz.; what is the value of a bar 6 in. long, and 1½ in. in breadth and thickness, a cubic inch weighing 131 oz.?

9. What is the cost of a marble slab, 6 ft. 3 in. long, 2 ft. 8 in. broad, 4 in. thick, at 14s. 6d. per cubic foot? What is the weight of the slab, one cubic foot weighing 170 lbs.?

10. The weight of a cubic inch of water is 253·17 grains, that of a cubic inch of air ·310017 grains; how many cubic inches of air are equal in weight to 1 cubic foot of water?

11. Find the cost of a beam of timber 24 feet long, 3 feet 6 inches broad, and 2 feet 5 inches thick, at 6s. 9½d. per cubic foot.

12. The bottom of a cistern measures 14 square feet; how deep should it be in order to contain 75 cubic feet of water?

13. If the bore of a certain pipe be 12 square inches, at what rate must water pass through it in order to supply 30,000 cubic feet in every twenty-four hours?

495. *VI. Spheres are to one another as the cubes of their diameters.*

The cubic content of a cylinder equals that of a rectangular solid having the same altitude, and whose base is equal to the area of the circle.

The cubic content of a sphere is equal to two-thirds of a cylinder having the same diameter and altitude.

*The cubic content of a cone is equal to one-third of a cylinder having the same diameter and altitude.**

496. TO FIND THE CONTENT OF A PARALLELOPIPED, A PRISM, OR A CYLINDER—

Multiply the number representing the area of the base by the number of linear units in the altitude.

497. TO FIND THE CONTENT OF A CONE—

Multiply the area of the base by one-third of the altitude.

498. TO FIND THE SOLIDITY OF A SPHERE—

Multiply the cube of the radius by 4·18879; or
Multiply the cube of the diameter by ·5236.

Note—The reason of these rules is to be found in the fact that 4·18879=two-thirds of 6·28318, which is the ratio of circumference to radius; and that ·5236 is equal to two-thirds of ·7854. (491.)

EXERCISE CXXXIV.

1. How many cubic inches are there in two spheres whose diameters are 17·725 ft. and 2·1 ft. respectively?

2. The circumference of a sphere is 125⅓ yards; find its solidity, also that of another sphere whose circumference is 24,900 miles.

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* These propositions are all demonstrated in any ordinary treatise on solid geometry. That part of Euclid's Elements which treats of this subject is seldom used.

3. The radius of the base of a cylinder is 56 ft. 8 in., and the height is half the circumference; required the solidity.

4. The length of a cylinder is 127½ ft., and the radius 13¾ ft.; what is the solidity?

5. What is the solidity of a cylinder whose length is 72·25 ft., and the circumference ¼ the length?

6. Required the cubic contents of a cylinder, having a length of 174·2 ft., and a circumference 17·42 ft.

7. The length of a hollow roller is 4 ft., exterior diameter 2 ft., and the thickness of the metal ¾ of an inch; determine its solidity.

8. If a cylinder, whose length is 13 ft. 4 in., contains 1,728 cubic feet, what length must be cut off so that it may contain one-fifth that number of cubic feet?

9. If a cylinder contains 692·5 cubic feet, its length being 74·25 ft., how long must it be to contain 138·5 cubic feet?

10. The diameter of a cone is 17 ft. 9 in., the perpendicular height twice the circumference; required the solidity.

11. Required the solidity of a cone whose perpendicular height is 25 ft 11 in., and the diameter 2 ft. 7 in.

12. The circumference of a cone is 25 ft., and the height four times the radius; required the solidity.

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1. How many paving stones, each 1 foot long and $\frac{4}{15}$ of a foot wide, will be required to pave a street 45 feet wide, surrounding a square, the side of which is 225 feet?

2. A wall, five times as high as it is broad, and 8 times as long as it is high, contains 18,225 cubic feet; what is its breadth?

3. In a rectangular court, which measures 96 feet by 84, there are four rectangular grass plots, measuring each 22½ feet by 18 feet; find the cost of paving the remainder of the court at 8½d. per square yard.

4. How many times is an arc of 57·2958 degrees contained in a semicircumference?

5. If in boring a circular well of 3 feet 5 inches diameter, 21 cubic yards of earth are dug out, how deep is the well?

PROGRESSION AND LOGARITHMS.

499. Any set or series of numbers of which each is related to its successor according to some fixed law said to be in Progression.

SECTION I.—PROGRESSION BY EQUAL DIFFERENCE, OR ARITHMETICAL PROGRESSION.

500. When any set or series of numbers is so arranged that each differs from that which precedes it by the same number, they are said to form an *Equi-different* or *Arithmetical Progression.*

The number by which they differ is called the Common Difference.

Let there be two sets of numbers—

I. 5 8 11 14 17 20 23 26 29 . . .

II. 55 50 45 40 35 30 25 20 15 . . .

In (I.) each number exceeds that which is before it by the constant difference 3. In (II.) each number is less than that which precedes it by the common difference 5. The former is an example of *ascending* and the other of *descending* progression.

501. *In an ascending progression the second term equals the first, plus the common difference; the third equals the first, plus twice the common difference; and generally, any term in the series is made up of the first term, plus the common difference multiplied by the number of terms before it.*

Observation.—If the word *minus* be substituted for *plus* throughout this proposition, it will obviously be true for all cases of descending progression.

502. *Demonstrative Example.*—If 3 be the first term, and 4 the common difference, then—

The second term $= 3 + 4$

The fifth term $= 3 + (4 \times 4)$

The ninth term $= 3 + (4 \times 8)$

The hundredth term $= 3 + (4 \times 99)$.

503. Suppose a, e, i, o, u are in ascending progression, and that d be their common difference,

Then $e = a + d$

$$i = e + d = a + d + d = a + 2d$$
$$o = i + d = a + 2d + d = a + 3d$$
$$u = o + d = a + 3d + d = a + 4d$$

If n be the number which denotes the place of a term, then $(n - 1)$ denotes how many terms stand before it.

General Formula.—Let a = first term, and d = common difference; then nth term $= a + (n - 1)d$.

If the progression be descending, the series is as follows—

$$e = a - d$$
$$i = e - d = a - d - d = a - 2d$$
$$o = i - d = a - 2d - d = a - 3d$$

and hence nth term $= a - (n - 1)d$.

504. TO FIND ANY GIVEN TERM OF AN ARITHMETICAL PROGRESSION—

RULE.

Multiply the common difference by the number of terms *minus one;* add this product to the first term if the progression be ascending, and subtract it from the first term if the progression be descending.

Example.—Find the 21st term of a progression whose first term is 5 and whose common difference is 8.

Here 21st term $= 5 + (20 \times 8) = 165$.

EXERCISE CXXXV.

☞ Find what the following terms are :—

1. First term 7, common difference 8; find the 9th term.
2. First term 6, common difference 12; find the 20th term.
3. First term 7, common difference 3; find the 12th term.
4. First term 3, common difference 2; find the 24th term.
5. First term 5, common difference 4; find the 100th term.
6. First term 6, common difference 8; find the 23rd term.
7. First term 11, common difference 2; find the 15th term.
8. First term 14, common difference 3; find the 23rd term.

9. First term 7, common difference 9 ; find the 11th term.
10. First term 1, common difference 11½ ; find the 5th term.
11. First term 20, second 17 ; find the 5th term.
12. First term 100, second 94 ; find the 7th term.
13. First term 53, second 51 ; find the 20th term.
14 First term 33, second 32½ ; find the 14th term.
15. First term 45, second 41·25 ; find the 6th term.

505. A method is easily deduced from this by which we may insert any number of terms between two numbers, so that the whole shall form an equi-different series. Terms thus inserted are called *differential* or *arithmetical means*.

For since the last term contains the first + the product of the common difference into the number of terms minus one,

506. TO INSERT ANY NUMBER OF ARITHMETICAL OR DIFFERENTIAL MEANS BETWEEN TWO NUMBERS—

RULE.

Subtract the less from the greater, and divide the remainder by the total number of the terms required to be inserted plus one. This will give the common difference.

Example.—Insert 9 differential means between 3 and 53.

Here, because there are in all 11 terms, 53 or the last term consists of the first term, plus 10 times the common difference.

$$\therefore \frac{53 - 3}{10} = 5 = \text{the common difference.}$$

$\therefore 3 + 5 =$ 2nd term, $3 + \overline{2 \times 5} =$ the third term, and the other terms proceed in the same order.

EXERCISE CXXXVI.

1. Insert 15 differential means between 5 and 37.
2. Insert 5 differential means between 8 and 36.
3. Insert 16 differential means between 5 and 56.
4. Insert 8 differential means between 6 and 600.

507. *If in a progression any two numbers be taken, equally distant from a third, their sum equals twice the third.*

This is shown in (54), for if in the following series—

6, 11, 16, 21, 26, 31, 36, 41, 46,

any one number, as 21, be taken, it is evident that this added to itself will equal the sum of 16 and 26, or of 11 and 31, or of 6 and 36, each of these pairs consisting of two numbers equally distant from 21, and one of each pair being as much less as the other is greater than the central number.

The number thus between the other two is their arithmetical mean or their average.

508. *Observation.*—The term *average* is also applied in cases when three, four, or many numbers are concerned. Thus, if there be three unequal numbers, and we add them together and take one-third of the sum, we find their average. So if there be seven unequal numbers, and we divide their sum by 7, we discover what seven equal numbers would amount to the same sum, and the number thus found is the average of the whole.

TO FIND THE ARITHMETICAL MEAN OR AVERAGE OF NUMBERS—

RULE.

Add them together, and divide their sum by the number of terms.

Example.—What is the arithmetical mean of 34 and 80?

Here $\frac{34+80}{2} = 57 =$ the arithmetical mean.

And 34, 57, and 80 form an arithmetical or equi-different progression.

EXERCISE CXXXVII.

☞ (a) Find the arithmetical means of the following pairs of numbers:

1. 72 and 36; 45 and 69; 83 and 57; 111 and 127.
2. 38 and 64; 25 and 135; 41 and 93; 61 and 85.
3. 100·2 and 7·8; 6·5 and 114·5; 8·4 and 11·26.
4. $\frac{3}{4}$ and 1·25; $\frac{7}{9}$ and $\frac{11}{16}$; $\frac{18}{26}$ and $\frac{6}{81}$.
5. $4\frac{1}{8}$ and $6\frac{2}{6}$; $27\frac{1}{9}$ and $35\frac{1}{12}$; 18 6 and 35·4.

(b) Find the average in the following sets of numbers:—

1. 18, 19, and 20; 15, 18, and 21; 23, 27, and 32.
2. 4, 5, 6, and 7; 8, 9, 10, and 11; 17, 23, 25, and 26.
3. 14, 25, 8, and 12; 4, 11, 13, and 8; 25, 16, 33, and 44
4. 79, 86, 53, and 12·5; 7·9, 9·9, and 8·9; 6·25, 3·1, and 4·7.

509. *In every arithmetical progression, the sum of any two terms at equal distance from the two extremes, equals the sum of the two extremes.*

Demonstrative Example I.—

2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35.

Here it follows from (55) that $2 + 35 = 5 + 32 = 8 + 29 = 11 + 26 = 14 + 23 = 17 + 20$.

General Formula.—If a, b, c, d, e, f, g be in arithmetical progression,

Then $a + g = b + f = c + e = 2d$.

Thus (55) every such series may be resolved into a number of pairs or couples, each having the same sum. And since the number of such pairs must always be half the number of terms, it follows that the sum of any two which form a pair, repeated as many times as there are pairs in the series, will give the sum of all the numbers in the series.

Demonstrative Example II.—

Let the series be 3, 7, 11, 15 · · · · · 35, 39, 43, 47
Then placing these numbers
in the reverse order, 47, 43, 39, 35 · · · · · 15, 11, 7, 3.

It is evident that the sum of every pair in the vertical lines is the same. Let the progression be of any length, $a, b, c \cdots\cdots x, y, z$ let the unknown sum be denoted by S, and let n represent the number of terms. If now $z, y, x \cdots\cdots c, b, a$ we write the same set of terms in the reverse order, the sum of the two series will be 2 S. But because (509) $a + z = b + y = c + x \cdots\cdots$ it follows that—

$2\,S = a + z$ taken as many times as there are terms.

Or $2\,S = (a + z)\,n$.

And if we divide both by 2, then $S = \dfrac{(a + z)\,n}{2}$.

General Formula.—If a = first term, z = the last term, and n the number of terms;

Then Sum of n terms $= (a + z) \times \dfrac{n}{2}$.

But since (502) $z = a + (n - 1)\,d$ when d is the difference,

Sum of n terms $= (a + a + \overline{n - 1}\;d)\,\dfrac{n}{2} = (2a + \overline{n - 1}\;d)\,\dfrac{n}{2}$.

510. TO FIND THE SUM OF AN EQUI-DIFFERENT SERIES—

RULE.

Find the last term by (504). Add the first and last terms together, and multiply by half the number of terms.

511. *Observation.*—Dividing the sum of the first and last terms by 2 gives us the *average* of the whole series; and n times the average is evidently the same as the sum of n unequal numbers.

Example.—What is the sum of 25 terms of the progression 2, 7, 12?

Here by (501) the 25th term $= 2 + \overline{5 \times 24} = 122$.

Therefore the sum $= \frac{(2 + 122)\,25}{2} = \frac{25 \times 124}{2} = 1550$.

EXERCISE CXXXVIII.

☞ Sum the following series—

1. 3, 9, 15, to 13 terms; and 1, 9, 17, to 100 terms.
2. 8, 15, 22, to 12 terms; and 1, 3, 5, to 25 terms.
3. 4, 7, 10, to 23 terms; and 12, 17, 22, to 86 terms.
4. 1, $1\frac{1}{2}$, 2, to 18 terms; and 5, $5\frac{1}{3}$, $5\frac{2}{3}$, to 21 terms.
5. 6, 6·25, 6·5, to 30 terms; and 11, 13·7, 16·4, to 15 terms.
6. 116, 108, 100, to 10 terms; and 270, 260, 250, to 11 terms.
7. $\frac{5}{7}$, 1, $1\frac{2}{7}$, to 8 terms; 6, $5\frac{1}{2}$, 5, to 25 terms.
8. 39, $35\frac{1}{2}$, 32, to 20 terms; and 1, $\frac{3}{2}$, 2, to 12 terms.
9. How many times does the hammer of a clock strike in a week?
10. What debt can be discharged in 2 years by monthly payments, at 6d. the first month and 8d. additional in each succeeding month?
11. A man travels 5 miles on one day, 8 the next, 11 the next, increasing his journey regularly at the same rate; in how many days will he travel 735 miles?
12. Add together all the numbers from 3 to 5000, both included.
13. A man receives a regular increase of 9d. a week in wages; if he begins at 10s. what will he earn a year and a half hence?
14. A falling body descends 16·1 feet in the first second, 48·3 in the second, and 80·5 in the third; how far would it fall in the seventeenth second of its descent?

512. *The sum of the series of odd numbers, beginning with unity, and carried to any given number of terms, always equals the square of that number.*

Demonstrative Example.—Let it be required to find the sum of 20 terms of the series 1, 3, 5, &c.

Here by (503) the 20th term $= 1 + \overline{19 \times 2} = 39$.

And by (510) Sum $= \frac{1 + 39}{2} \times 20$.

But $\frac{1 + 39}{2} = 20$; $\therefore$ Sum $= 20 \times 20 = 400$.

In the same manner it might be shown that in this series the sum of 27 terms $= 27^2$; the sum of 100 terms will be 100^2, &c.*

General Formula.—If $a = 1$ and $d = 2$,

Then Sum $= \frac{(2 + (n - 1)\,2)\,n}{2} = \frac{(2 + 2n - 2)\,n}{2} = n^2$.

513. *Corollary.—If there be any arithmetical series such that th, common difference is double the first term, the sum of* n *terms of that series equals the product of the first term and* n^2.

Example.—16, 48, 80, 112, &c.

Here first term = 16, and common difference 32, or 2 × 16.

Hence the sum of 7 terms $= 16 \times 7^2$.

This progression represents the number of feet through which a falling body passes in successive seconds, and the rule is therefore of importance. To find the space traversed in n seconds, multiply 16 by n^2.

EXERCISE CXXXIX.

1. Find the sums of 24, and of 93 terms of the series of odd numbers.
2. Find the sum of 25 terms of the series 3, 9, 15, &c.
3. Find the sum of 237 terms of the series 2, 6, 10, &c.
4. Find the sum of 185 terms of the series 5, 15, 25, &c.
5. Find the sum of 25 terms of the series 3·5, 10·5, 17·5, &c.
6. Find the sum of 17 terms of the series 12, 36, 60, &c.
7. How far will a body fall in 19 seconds?

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* The student should compare this truth with that given in (429) and endeavour to detect the same principle under both forms.

SECTION II.—PROGRESSION BY EQUAL RATIOS, OR GEOMETRICAL PROGRESSION.

514. When the numbers in any series are in continued proportion (324), or are so arranged that each is formed from that which precedes it by the same multiplier, they are said to be in *Geometrical** or *Equi-rational Progression.*

The constant multiplier is called the Ratio of the progression. The progression is either ascending or descending, according as the constant multiplier or ratio is greater or less than unity.

Example I.— $4 : 12 : 36 : 108 : 324 : 972.$

Example II.— $36 : 12 : 4 : \frac{4}{3} : \frac{4}{9} : \frac{4}{27}.$

In (*I.*) the common ratio is 3, and the series is an *ascending* one.

In (*II.*) the common ration is $\frac{1}{3}$, and the series is a *descending* one.

These series are in continued proportion (324), and each term is called a *proportional mean* between the two adjacent terms.

515. *Observation.*—It follows from this arrangement—1. That any extremes equals the square of a proportion of which the product of the three consecutive terms form the mean (324). 2. That any four consecutive terms form a proportion of which the product of the extremes equals the product of the means (319). 3. That every term is both the antecedent and the consequent of a ratio, except the first, which is only an antecedent, and the last, which is only a consequent. 4. That every truth enunciated concerning proportions applies to such a series in a great variety of ways.

For example, there needs no new demonstration to prove that—1. If numbers be in geometrical progression their differences are also in geometrical progression; and 2. That the first term is to the third as the square of the first is to the square of the third (326) and (331).

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* The term *Geometrical* is obviously inappropriate here, but it is so universally used that we have thought it inexpedient to abandon it. The discussion in the foot-note on page 240 will serve to account for the use of the term. We measure surfaces by taking the *product* of the numbers representing their dimensions, and hence that part of arithmetic which treats of the products of numbers has become encumbered with geometrical terms. There can be no great harm in employing them if the limitation of their meaning be properly understood, but otherwise they tend to mislead a student, and should be used with great caution.

EXERCISE CXL.

☞ Construct 12 series of numbers in geometrical progression, viz.—
Six containing 20 terms, and having a ratio greater than unity;
And Six containing 15 terms, and having ratios less than unity.

516. *In any geometrical progression, the second term equals the first, multiplied by the common ratio; the third equals the first, multiplied by the square of the common ratio; and any term equals the first, multiplied by that power of the common ratio which is indicated by the number of terms which are before it in the series.*

Demonstrative Example.—Let 4 be the first term and 5 the common ratio.

Then 1st term $= 4$
2nd term $= 4 \times 5$
3rd term $= 4 \times 5 \times 5 = 4 \times 5^2$
4th term $= 4 \times 5^2 \times 5 = 4 \times 5^3$
5th term $= 4 \times 5^3 \times 5 = 4 \times 5^4$
. . . .
20th term $= 4 \times 5^{19}$

General Formula.—If $a =$ 1st term, and $r =$ common ratio;
Because 1st term $= a$; 2nd term $= ar$; 3rd term $= ar^2$;
$\therefore$ nth term $= ar^{n-1}$.

517. TO FIND ANY TERM OF A GEOMETRICAL PROGRESSION—

RULE.

Multiply the first term by the common ratio, raised to a power indicated by one less than the number of terms.

Example I.—Find the fifth term of the series 2, 6, &c.
Here $2 =$ first term, and $3 =$ common ratio.
$\therefore$ 5th term $= 2 \times 3^{5-1} = 2 \times 3^4 = 2 \times 81 = 162$.

Example II.—Find the ninth term of the series 12, 6, 3, &c.
Here 1st term $= 12$, and common ratio $= \frac{1}{2}$.
$\therefore$ 9th term $= 12 \times (\frac{1}{2})^{9-1} = 12 \times (\frac{1}{2})^8 = 12 \times \frac{1}{256} = \frac{12}{256}$.

EXERCISE CXLI.

1. Find the fifteenth term of the series 3, 6, 12, &c.
2. Find the twelfth term of the series 1, 2, 4, &c.

3. Find the eighth term of the series 2, 6, 18, &c.
4. Find the fifth term of the series 7, 28, 112, &c.
5. Find the eleventh term of the series 6, 3, $\frac{3}{2}$, &c.
6. Find the eighth term of the series 5, 20, 80, &c.
7. Find the seventh term of the series, 3, 30, 300, &c.
8. Find the sixth term of the series 2, $\frac{2}{3}$, $\frac{2}{9}$, &c.
9. Find the twelfth term of the series 2, 6, 18, &c.
10. Fine the tenth term of the series 12, 6, 3, &c.
11. The first term of a geometrical series is 5, and the second 12½, find the seventh.
12. What is the twentieth term of the series 1, 2, 4, &c.?
13. What is the ninth term of the series 1, ·5, ·25, &c.?
14. Find the seventh term of the series 8, 4, 2, &c.
15. What is the fifth term of the series 9, 6, 4, &c.?
16. Find the tenth term of the series $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$, &c.
17. Find the eighth term of the series $1\frac{4}{5}$, 1·2, ·8, $\frac{8}{15}$, &c.

518. *The square root of the product of two numbers is always their proportional mean.*

Demonstrative Example.—Let a proportional mean be required between 2 and 200.

Here by (324) if 2, x, and 200 be in proportion, $2 \times 200 = x^2$.

And because $2 \times 200 =$ the square of the mean;

$\therefore$ the mean $= \sqrt{2 \times 200} = \sqrt{400} = 20$.

$\therefore 2 : 20 : 200$ are in geometrical progression, and 20 is the proportional mean.

General Formula.—$a : \sqrt{ab} : b$.

TO FIND A PROPORTIONAL MEAN BETWEEN TWO NUMBERS—

RULE.

519. Find their product and extract its square root.

EXERCISE CXLII.

☞ Find a mean proportional between the following numbers :—

1.	18 and 162;	29 and 1421;	6 and 54.
2.	15 and 240;	18 and 450;	13 and 1573.
3.	2 and 18;	35 and 46;	279·2 and 3472·9.

520. To insert any number of proportional means between any two numbers it is only necessary to find the common ratio. But to do this often involves the extraction of higher roots. Thus, if the first term and the fourth are given, and we are required to find the second and third, we must seek the common ratio.

But since the 4th term = the first term × the cube of the common ratio, we must divide the fourth term by the first, and then extract the third root of the quotient in order to find a common ratio. Similarly, if we have to insert 7 proportional means between two numbers, these two numbers will form the 1st and 9th terms of the series. But the 9th term is made up of the product of the first and the 8th power of the common ratio. Here, therefore, it would be necessary to divide the greater number by the less, and to extract the 8th root of the quotient; this would give the common ratio.

General Formula.—If r be a common ratio and n the number of terms, a the first term and z the last;

$$\text{Then } r = \sqrt[n-1]{\frac{z}{a}}.$$

521. It will thus be seen that the common ratio can be obtained by ordinary arithmetic, only when the number representing the root to be found is either 2 or 3. But by logarithms it will be seen (550) that questions of this kind can be solved.

522. *In every geometrical progression, the product of any two terms equally distant from the two extremes is the same as the product of the extremes.*

Demonstrative Example.—5 : 15 : 45 : 135 : 405 : 1215 : 3645.

Here because 15 is as many times more than 5 as 1215 is less than 3645 ∴ (126) $15 \times 1215 = 5 \times 3645$.

And because 45 is as many times more than 5 as 405 is less than 3645 ∴ (126) $45 \times 405 = 5 \times 3645$.

And because 135 is as many times more than 5 as it is less than 3645 ∴ (124) $135^2 = 5 \times 3645$.

Hence $5 \times 3645 = 15 \times 1215 = 45 \times 405 = 135^2$.

General Formula.—If $a : b : c : d : e : f : g$;

Then $ag = bf = ce = d^2$.

Hence (127) *every geometric series may be resolved into a number of pairs of factors, each pair having the same product as the square of the middle term, or as the product of the two extremes.*

523. This truth does not, however, like the analogous truth in Arithmetical Progression (508) enable us to find the *sum* of the series;* it is necessary, therefore, to seek some other method of arriving at this result.

Let the series whose sum is required be—

2, 6, 18, 54, 162, 486, 1458, and let S = their sum.

Here the common ratio is 3. Now if we multiply every term of the series by the common ratio, we find that—

I. 3 S = 6 + 18 + 54 + 162 + 486 + 1458 + 4374.
II. But S = 2 + 6 + 18 + 54 + 162 + 486 + 1458.

Here the same set of numbers occurs in both series, except the last term of (I) and the first of (II.). Therefore subtracting (II.) from (I.) we have 2 S = 4374 − 2 = 4372; and S = 2187.

Or let the series be $a, ar, ar^2, ar^3, ar^4, \ldots\ldots ar^{n-1}$.

Then $S = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots ar^{n-1}$. (1)

And $Sr = ar + ar^2 + ar^3 + ar^4 + ar^5 \ldots\ldots ar^n$. (2)

Subtract (1) from (2), then $S(r - 1) = ar^n - a$.

General Formula.—$S = \frac{ar^n - a}{r - 1} = \frac{(r^n - 1)a}{r - 1}$.

In the case in which the common ratio was 3, and we multiplied the last term by 3 and subtracted the first term, the result gave *twice* the sum. In like manner, had the common ratio been 6, six times the last term, minus the first term, would have given *five* times the sum. In (266) this principle was applied to the summation of recurring decimals, and then, as 10 or some power of 10 was always the common ratio, the result of the same operation always gave 9 or 99 or 999, &c., times the sum.

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* If the inference were of any value, it would be interesting to notice that we might here obtain an estimate of the *product* of all the terms of a geometric series, by a method analogous to that by which we obtained the *sum* of an arithmetical series. For it follows from what is said in the text, that *the product of all the terms of a geometric progression, consisting of* n *terms, equals the square root of the* n*th power of the product of the two extremes.* But such a proposition is never needed in practice.

524. TO FIND THE SUM OF A GEOMETRIC PROGRESSION—

Find the last term by (517). Multiply this by the common ratio and subtract the first term. This answer divided by a number *one less* than the common ratio will give the sum.

Example I.—Find the sum of 7 terms of the series 2, 4, 8, &c.

Here because nth term $= ar^{n-1}$, and $r = 2$, and $n = 7$;

$$\therefore \text{(516)} \quad \text{7th term} = 2 \times 2^6 = 128.$$

And because $S = \dfrac{(n\text{th term} \times r) - a}{r - 1}$;

$$\therefore \text{Sum of 7 terms} = (128 \times 2) - 2 = 254.$$

Example II.—Find the sum of 5 terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, &c.

Here the common ratio is $\frac{2}{3}$.

$$\therefore \text{the 5th term} = (\tfrac{2}{3})^4 = \tfrac{16}{81}.$$

$$\therefore S = \frac{(\frac{16}{81} \times \frac{2}{3}) - 1}{\frac{2}{3} - 1} = \frac{-\frac{211}{243}}{-\frac{1}{3}} = 2\tfrac{49}{81}.$$

Example III.—Sum the series ·43434343, *ad infinitum.*

This is equivalent to $\frac{43}{100} + \frac{43}{100^2} + \frac{43}{100^3} + \frac{43}{100^4}$, &c., and is therefore a series of progression by equal ratios, the common ratio being $\frac{1}{100}$.

Multiply every term by 100.

$$\text{Then } 100\,S = 43 + \frac{43}{100} + \frac{43}{100^2} + \frac{43}{100^3} + \frac{43}{100^4}, \text{ \&c.}$$

$$\text{Subtract } S = \frac{43}{100} + \frac{43}{100^2} + \frac{43}{100^3} + \frac{43}{100^4}, \text{ \&c.}$$

Then 99 S = 43. Therefore $S = \frac{43}{99}$.

Example IV.—Sum the series $\frac{1}{4} + \frac{1}{16}, + \frac{1}{64}$, *ad infinitum.*

Here $\frac{1}{4}$ is the common ratio, we therefore multiply every term by 4.

$$\therefore \quad 4\,S = 1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} + \text{\&c., } ad\ inf$$

$$\text{Subtract } S = \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{64} + \text{\&c., } ad\ inf.$$

$$3\,S = 1 \quad \text{and } S = \tfrac{1}{3}.$$

EXERCISE CXLIII.

☞ Sum the following series :—

1. 1, 4, 16, &c., to 7 terms, and 3, $4\frac{1}{2}$, $6\frac{3}{4}$, to 5 terms.
2. 4, 3, $\frac{9}{4}$, to 10 terms, and $\frac{3}{2}$, 1, $\frac{2}{3}$. to 12 terms.
3. 5, 20, 80, to 8 terms, and 100, 40, 16, to 10 terms.
4. $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{8} + \frac{1}{16}$, &c., *ad inf.*, and $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, *ad inf.*
5. Suppose a ball to be put in motion by a force which drives it 12 miles the first hour, 10 miles the second, and so on, continually decreasing in the proportion of 12 to 10 to infinity; what space would it move through?
6. If one grain of wheat be placed on the first square of a chess-board, two on the second, four on the third, and so on, doubling the number for each of the 64 squares, how many bushels of wheat would be required, supposing that 7,680 grains fill a pint measure?
7. Sum the series 5, 20, 80, &c., to 8 terms.

525. We may trace a certain correspondence between the analogous truths relating to the two kinds of progression. Thus—

I. Arithmetical Progression.	*II. Geometrical Progression.*
In which a = first term, d = common difference, and n = number of terms.	In which a = first term, r = common ratio, and n = number of terms.
(503) nth term $= a + \overline{n - 1}\, d$.	(516) nth term $= a \times r^{n-1}$.
And $d = \dfrac{n\text{th term} - a}{n - 1}$.	(520) $r = \sqrt[n-1]{\dfrac{n\text{th term}}{a}}$.
By (510), sum of all the terms $= \dfrac{(a + n\text{th term})\, n}{2}$.	By (522) product of all the terms $= \sqrt{(a \times n\text{th term})^n}$.

On comparing these two we see that—

Addition in (I.) corresponds to Multiplication in (II.).

Subtraction in (I.) corresponds to Division in (II.).

Multiplication by a given number in (I.) corresponds to Involution to a given power in (II.).

Division by a given number in (I.) corresponds to Evolution of a given root in (II.).

SECTION III.—LOGARITHMS.

526. The resemblances noticed in the last paragraph are of great importance in Arithmetic. They probably suggested to the mind of the inventor * the method of calculating by Logarithms, or of establishing such a connection between numbers in arithmetical and others in geometrical progression, as shall enable us to deal with the latter by means of the simpler operations of the former, and substitute addition and subtraction in long and involved computations for multiplication and division. The one arithmetical truth on which the use of this method is founded, and which must be borne in mind throughout this chapter, is as follows.

527. *We multiply different powers of any number together when we add the exponents of those powers.*

Demonstrative Example.—

$$5^3 \times 5^6 = (5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5 \times 5 \times 5) = 5^9.$$

$$a^2 \times a^5 = aa \times aaaaa = a^{2+5} = a^7.$$

We divide one power of a number by another power of the same number when we subtract the exponent of the divisor from that of the dividend.

Demonstrative Example.—

$$20^7 \div 20^3 = \frac{20 \times 20 \times 20 \times 20 \times 20 \times 20 \times 20}{20 \times 20 \times 20} =$$

$$20 \times 20 \times 20 \times 20 = 20^4 \text{ or } 20^{7-3}.$$

$$a^9 \div a^6 = \frac{aaaaaaaaa}{aaaaaa} = aaa = a^3 = a^{9-6}.$$

528. Let us now form the simplest series in arithmetical progression, beginning with 0, and having 1 for the common difference;

And also a simple geometric progression, beginning with 1, and having 3 for the common ratio.

I.	0	1	2	3	4	5	6	7	8	9	10	11.
II	1	3	9	27	81	243	729	2187	6561	19683	59049	177147.

Here the numbers in the upper line are called the *logarithms* of those which are placed beneath them.

Observation.—Logarithm, from λόγων ἀριθμός (*logōn arithmos*), = the number of the ratios. Thus, 7 is the logarithm of 2187, because it represents the *number of the ratios* or powers of three which are contained in 2187, and 10 is the logarithm of 59049.

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* Baron Napier, or Neper, of Murchiston, in Scotland, published a book describing his discovery in the year 1614. The book was entitled "Mirifici Logarithmorum Canonis Descriptio"—An Account of the Marvellous System of Logarithms.

529. The exponent in any numerical expression may therefore be considered as a logarithm.

Thus $a^x = b$. Here x is the logarithm of b to the base a, or it expresses the *number of the ratios* or powers of a which are contained in b.

It is usual to contract the expression thus, log. b means logarithm of b.

530. Suppose it be required to multiply two numbers in the lower series together, 243 and 729 for instance; we notice that above them are the numbers 5 and 6. Now $5 + 6 = 11$, and under 11 we may find 177147, or the product of 243 and 729.

This would evidently be the case however far the series might be extended, for all the numbers in the lower line represent powers of 3, and the figures above them are the exponents of those powers.

And because $243 = 3^5$, and $729 = 3^6$;

$$\therefore 243 \times 729, \text{ or } 3^5 \times 3^6 = 3^{11} = 177147.$$

531. In like manner we may divide any one of the terms in the lower series by any other which is less than itself, if we subtract the exponent of the divisor from that of the dividend, and take the number which stands underneath the difference.

Thus because $59049 = 3^{10}$, and $729 = 3^6$;

$$\therefore \text{by } (527)\ \frac{3^{10}}{3^6} = 3^4 = 81.$$

532. We may construct any other series fulfilling the conditions described in 528, and obtain similar results; *e g.*,

0	3	6	9	12	15	18	21	24	27	30	33	36	39.
1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192

Take any two numbers from the lower series,—32 and 128. Now find a number as many places to the right of 128 as 1 is to the left of 32.

Then $1 : 32 :: 128 : 4096$.

Now by the law of proportion (319)

$$1 \times 4096 = 32 \times 128.$$

But above these numbers are 0, 15, 21, 36, of which the second exceeds the first as much as the fourth exceeds the third.

And by the law of arithmetical progression (55)

$$0 + 36 = 15 + 21.$$

But 36 = log. 4096, and 15 = log. 32, and 21 = log. 128.

Hence log. (32 × 128) = log. 32 + log. 128.

533. The following are obvious inferences from this reasoning :—

I. The sum of the logarithms of two or more numbers equals the logarithm of their product.

General Formula.—Log. abc = log. a + log. b + log. c.

And log. $a^n = n \times$ log. a.

II. The difference between the logarithms of two numbers equals the logarithm of their quotient.

General Formula.—Log. $\frac{a}{b}$ = log. a − log. b, and log. $\sqrt[n]{a} = \frac{\text{log. } a}{n}$.

534. These relations once established between any two series of numbers, it follows that, as far as those series extend, we may be spared the trouble of multiplying and dividing numbers whose logarithms are known, and may add or subtract those logarithms instead.

It thus becomes desirable to construct a table which shall comprise all ordinary numbers, and shall give the logarithms of each. For this purpose any number may be chosen as the base ; *e. g.*,

I.	0	1	2	3	4	5	6	7	8	9.
	1	2	4	8	16	32	64	128	256	512.

II.	0	1	2	3	4	5	6	7.
	1	5	25	125	625	3125	15625	48625.

Of these systems, (I.) is constructed with the base 2, and (II.) is constructed with the base 5.

535. *Observation.*—No use can be made in calculation of such tables as these, however far they may be carried ; because the numbers whose logarithms are given are very few, and the tables contain no provision for dealing with any numbers unless they happen to be expressed with integral exponents as powers of the base. It is necessary, therefore, whatever system be adopted, to calculate the fractional exponents of all the intermediate numbers.

536. Two or three important facts require to be noticed here.

I. The logarithm is, in every case, the *differential* mean between the two logarithms, one on each side of itself.

II. The number in the lower line is, in every case, the *proportional* mean between the two numbers, one on each side of itself.

Thus, in series (II.), 3, 4, and 5 are in equi-different progression, while 125, 625, 3125, are in equi-rational progression.

∴ by (508) $\frac{3 + 5}{2} = 4$ = the *differential* mean between 3 and 5.

And by (510) $\sqrt{3125 \times 125} = 625$ = the *proportional* mean between 125 and 3125.

537. In like manner, if any two terms be taken at random in the series of logarithms, the other logarithms between them form a series of differential means; while the numbers to which the logarithms belong form a series of proportional means between the two extremes.

Thus, in (I.), we take from the upper line 3 and 8.

Now the logarithms between these, viz., 4, 5, 6, and 7, are 4 differential means between 3 and 8 (505).

But the numbers of which 3 and 8 are logarithms are 8 and 256.

And the series of numbers between them is 16, 32, 64, and 128.

And these are four proportional means between 8 and 256.

Hence we infer that—

538. *If we take any two numbers whose logarithms are known, and find a mean proportional between them, the result will be a number whose logarithm will be a differential mean between the two logarithms.*

For the differential mean between 2 and 3 is $2\frac{1}{2}$, or 2·5.

Hence in (I.) 2·5 is the log. of $\sqrt{4 \times 8}$, or of the proportional mean of those numbers whose logs. are 2 and 3.

Also in (II.) 2·5 is the log. of $\sqrt{25 \times 125}$ for the same reason.

539. The system of logarithms now universally adopted is the decimal.* It was invented by Briggs, and is commonly called his system; it is as follows:—

0	1	2	3	4	5	6	7	8	&c
1	10	100	1000	10000	100000	1000000	10000000	100000000,	&c

In order to construct the common tables on this basis, it becomes necessary *to fill up the lower of these lines with the set of natural numbers; to consider each of them as a term in geometrical proportion; and to connect each with another number, which shall show what term it is in the series.* The process of discovering the logarithms of the intermediate numbers is a very difficult and laborious one, involving the use of some of the most advanced truths in mathematical science. The logarithms of any numbers which are not powers of 10 are surds, and can only be approximately expressed.

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* The inventor of logarithms was led to his discovery by the desire to simplify geometrical and astronomical formulæ. He did not fully anticipate their extensive application to ordinary arithmetical processes. Hence the *radix* or *base* of his scale was determined for certain geometrical reasons. We cannot enter into them in this place, but it is worth remembering that in the Napierian or Hyperbolic system the logarithm of 10 is 2·3025852; the number whose logarithm is 1 being 2·7182818. This latter number is called the *base* of the Hyperbolic or Napierian Logarithms; and it is evident that any logarithm on the decimal system, multiplied by the former number will give the logarithm on the Hyperbolic. Henry Briggs published, in the year 1617, the book explaining the change which he proposed to make in the base.

540. Suppose now that by the higher processes we have referred to, it be found that the logarithm of 5 may be approximately expressed thus—

$$\text{Log. } 5 = {\cdot}69897.$$

This means that $10^{\cdot 69897} = 5$, or that ·69897 is the exponent which represents what power of 10 the number 5 is.

We may obtain many inferences from this fact; *e. g.*,

I. From (533) it follows that—

Log. $5 \times 5 =$ log. $5 +$ log. $5 = 2$ log. 5.

Or log. $25 = 2 \times {\cdot}69897 = 1{\cdot}39794$.

II. And log. $10 \div 5 =$ log. $10 -$ log. 5.

But log. $10 = 1$, $\therefore$ log $2 = 1 - {\cdot}69897 = {\cdot}30103$.

III. Because $8 = 2 \times 2 \times 2$, $\therefore$ log. $8 = 3$ log. 2.

$\therefore$ from (II.) log. $8 = 3 \times {\cdot}30103 = {\cdot}90309$.

IV. And log. $16 =$ log. $2 +$ log. $8 = {\cdot}30103 + {\cdot}90309 = 1{\cdot}20412$.

V. Log. $\sqrt{16}$ or log. $4 = \dfrac{\text{log. } 16}{2} = \dfrac{1{\cdot}20412}{2} = {\cdot}60206$.

VI. And log. 5^7 or log. $48625 = 7$ log. $5 = {\cdot}69897 \times 7 = 4{\cdot}87479$.

541. Because the logarithms of the several powers of 10 are integer numbers, it follows that if the logarithm of a number be known, the logarithm of that number, multiplied or divided by any power of 10, will differ from it in the whole number, not in the fractional part. The whole number is called the *characteristic*, and the fractional part the *mantissa*.

Thus because— log. $25 = 1{\cdot}39794$

Log. $250 =$ log. $10 +$ log. $25 = 2{\cdot}39794$

Log. $2500 =$ log. $100 +$ log. $25 = 3{\cdot}39794$

Log. $25000 =$ log. $1000 +$ log. $25 = 4{\cdot}39794$

Hence also log. $25 \div 10$, or log. $2{\cdot}5 = {\cdot}39794$

Log. $25 \div 100$, or log. ${\cdot}25 = {\cdot}39794 - 1$, or $\bar{1}{\cdot}39794$

Log. $25 \div 1000$, or log. ${\cdot}025 = {\cdot}39794 - 2$, or $\underline{2}{\cdot}39794$

Log. $25 \div 10000$, or log. ${\cdot}0025 = {\cdot}39794 - 3$, or $\underline{3}{\cdot}39794$

Log. $25 \div 100000$, or log. ${\cdot}00025 = {\cdot}39794 - 4$, or $4{\cdot}39794$

Observation.—On referring to a book of tables to find the logarithm either of 25, or 25000, or ·025, or any of the numbers just given, the only figures found would be 39794. The characteristic is never given, and is determined by the following considerations:—

542. I. From (539) it will be seen that the characteristic of every number between 1 and 10 is 0, that of every number between 10 and 100 is 1, that of every number between 100 and 1000 is 2; and generally,

The characteristic of every integer number is the same as the number of its digits, minus one.

543. II. The *characteristic* of a decimal fraction or of a number divided by 10, or some power of 10, must always be subtracted, and is written with the sign of subtraction over it, thus: "log. ·0025 = $\bar{3}$·39794" means that though the mantissa may be added, the characteristic must be subtracted from any other logarithm which may require to be connected with it. If the first significant figure of the number be in the first decimal place, the characteristic is $\bar{1}$; if it be in the second, the characteristic is $\bar{2}$; and generally,

The characteristic of any decimal fraction has the minus sign above it, and is the number which represents the place of the first significant figure to the right of the decimal point.

544. Since (533) the logarithm of any composite number can readily be deduced from the logarithms of its factors, if the logarithms of prime numbers be once ascertained, those of all other numbers can readily be found. The following is a table of the logarithms of all the prime numbers up to 100, expressed as in ordinary books of tables.

Prime Numbers.	Logarithms.	Prime Numbers.	Logarithms.
2	30103	43	6334685
3	4771213	47	6720979
5	69897	53	7242759
7	845098	59	770852
11	0413927	61	7853298
13	1139434	67	8260748
17	2304489	71	8512583
19	2787536	73	8633229
23	3617278	79	8976271
29	462398	83	9190781
31	4913617	89	94939
37	5682017	97	9867717
41	6127839		

545. TO FIND THE LOGARITHM OF A NUMBER—

RULE.

Resolve the number into its prime factors, and add the logarithms of those factors together.

Or, if the number be a fraction, subtract the logarithm of the denominator from that of the numerator, and the remainder is the logarithm of the quotient or of the fraction.

Or, if the number be a power of a known number, multiply the logarithm of that known number by the exponent of the power.

Or, if the number be a root of a known number, divide the logarithm of that known number by the index or exponent of the root.

Example.—From the logs. of 2 and 7 deduce the logs. of 2800, of 3·43, of $12\frac{1}{4}$, and of $\sqrt[3]{14}$.

1. Because $2800 = 2 \times 2 \times 7 \times 100$,
$\therefore$ log. 2800 = log. 2 + log. 2 + log. 7 + log. 100.
$= {\cdot}30103 + {\cdot}30103 + {\cdot}845098 + 2 = 3{\cdot}447158$.

2. Because $3{\cdot}43 = 7 \times 7 \times 7 \times \frac{1}{100}$,
$\therefore$ log. 3·43 = 3 log. 7 − log. 100.
$= 3 \times {\cdot}845098 - 2 = {\cdot}5352941$.

3. Because $12\frac{1}{4} = \frac{49}{4}$, $\therefore$ log. $12\frac{1}{4}$ = log. 49 − log. 4.
But log. 49 = 2 log. 7 = 1·690196, and log. 4 = 2 log. 2 = ·60206.
$\therefore$ log. 12·25 = 1·690196 − ·60206 = 1·088136.

4. Because $\sqrt[3]{14} = \sqrt[3]{2 \times 7}$, $\therefore$ log. $\sqrt[3]{14} = \dfrac{\text{log. } 2 + \text{log. } 7}{3}$.
$= ({\cdot}30103 + {\cdot}845098) \div 3 = {\cdot}3820426$.

EXERCISE CXLIV.

1. From logs. 2 and 3 deduce the logs. of the numbers 6, 27, 3·6, 1600, 6·4, 2·25, $\sqrt{2}$, 12, $\cdot\dot{3}$, and $\cdot\dot{8}$.

2. From logs. 5 and 7 deduce the logs. of ·35, of 49000, of 6.25, of 17·5, of $\frac{14}{25}$, of $1\frac{3}{7}$, of 2450000.

3. From logs. 7 and 2 deduce those of 1400, of 5·6, of ·00196, and of 980.

4. Given logs. 3 and 2, deduce from them those of 60, of 27000, of 1620, of 405, of 3·6, and of 4800.

5. Given logs. 5, 11, and 3, deduce logs. 605, 1665, ·0045, of $\frac{1}{15}$, of 729, of 91·6, of 44, of 6·75, of 198, and of 24·2.

6. From logs. 5 and 3 deduce the logs. of 450, of 6, and of $\sqrt{\frac{3}{5}}$.

7. With the help of the table find the log. of the result of this fraction. $\frac{23^2 \times 15^4}{19^6}$.

8. From the table deduce the log. of the fraction, $\sqrt[3]{\frac{12^3 \times 5^8}{37^2}}$.

9. Complete the table, so that it shall contain the logs. of all numbers between 1 and 100, with their characteristics.

546. *Calculation of Differences.**—Ordinary tables only give the logarithms of numbers consisting either of 4 or 5 digits, the logarithms themselves being generally carried to the sixth or seventh place. If we desire to find the logarithm 254329, and on looking into a table find that the logarithms of 2543 and of 2544 are given, but not of any intermediate numbers, the number must be calculated in the following manner:—

Log. 2543 = ·40535, and log. 2544 = ·40552.

The difference between these logarithms is 17.

Now the difference between 254300 and 254400 is 100, and the difference between 254300 and 254329 is 29. Hence we have the following proportion:—*If for* 100, *difference between* 254300 *and* 254400, *we have* 17 *difference between their logarithms, what should be the difference between the logarithms of the two numbers,* 254300 *and* 254329, *which differ by* 29?

The proportion is $100 : 29 :: 17 : x$. And $x = \frac{29 \times 17}{100} = 4·93$.

Hence log. 254329 = 5·40535 + ·0000493 = 5·4054.

547. *Observation.*—The difference is always given in a separate column in the tables, and is called the tabular difference. The proportion thus established between the differences of numbers and the differences of their logarithms is not strictly true, but gives a result sufficiently near for all practical purposes.

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* Throughout the remainder of this section it is assumed that the student has a book of tables at hand. Hutton's and Babbage's are the completest; but any set will answer the purpose.

548. When the logarithm of the answer to a question is found it becomes necessary to find the number of which it is the logarithm. To do this we must look in the table for the number, omitting the characteristic. For example : if the answer to a sum were to be 2·3991717 we should look in the table for ·3991717, and having found the number 25071 as the natural number belonging to it, we should then use the characteristic 2, in order to determine the value of these figures. Now, since 2·3991717 must by (541) be the logarithm of a number between 100 and 1000, the decimal point must be placed after the third figure, and 2·3991717 = log. 250·71.

In this case the given logarithm was found exactly in the tables; but this does not always happen, and it often becomes necessary, therefore, to calculate the difference. Thus : suppose it is required to find the number corresponding to 6·45936; we look in a table and find that log. 2879 = 45924, and log. 288 = 45939, the tabular difference being 15 : we may then make this proposition—*If for* 15 *difference between the logarithms of* 2879 *and* 2880 *there be a difference of one in the numbers themselves, what should be that number whose logarithm differs from that of* 2879 *by* 11 ?

$$15 : 11 :: 1 : x, \quad \text{and } x = \tfrac{11}{15} = \cdot 8.$$

Hence 45936 = log. 28798, and 6·45936 = log. 2879800.

APPLICATION OF LOGARITHMS.—I. MULTIPLICATION, ETC.

The following examples, worked by the help of logarithmic tables, will show how they are applicable to involved problems in arithmetic :—

549. *Example I.*—Simplify the expression $\dfrac{472{\cdot}8^3 \times \sqrt[5]{268450}}{1876 \times 327^2}$.

Now by (545) the logarithm of the answer to this sum will be—

$$3 \text{ log. } 472\,8 + \frac{\text{log. } 268450}{5} - (\text{log. } 1876 + 2 \text{ log. } 327).$$

But log. 472·8 = 2·6746775. Hence log. $472{\cdot}8^3$ = 8·0240325.
And log. 268450 = 5·4288634. Hence log. $\sqrt[5]{268450}$ = 1·08577268
Logarithm of numerator = 9·10980518
Log. 1876 3·2732328
2 log. 327 = 2 × 2·5145478 = 5·0290956
Logarithm of denominator = 8·3023284 = 8·3023284
Log. 64191 ·80747678

But since the characteristic of the logarithm is 0, the answer lies between 1 and 10, and is therefore 6·4191.

550. By logarithms we may readily solve those questions in Geometrical Progression (521) for which the ordinary processes of Evolution will not suffice.

Example II.—Suppose it be required to insert 6 proportional means between 2 and 4374. Here it is first necessary to find the common ratio of the series, which consists of 8 terms.

By (520) $r = \sqrt[n-1]{\frac{n\text{th term}}{a}}$ $\therefore r = \sqrt[7]{\frac{4374}{2}} = \sqrt[7]{2187}.$

$\therefore \log.\ r = \log.\ 2187 \div 7 = \frac{3{\cdot}3398488}{7} = {\cdot}477121 = \log.\ 3.$

Hence 3 is the common ratio, and 6, 18, 54, 162, 486, 1458, form the series required.

Example III.—Insert 25 proportional means between 3 and 4.

Here $r = \sqrt[26]{\frac{4}{3}}$, and $\log.\ r = \frac{\log.\ 4 - \log.\ 3}{26}$.

But $\log.\ 4 - \log.\ 3 = {\cdot}60206 - {\cdot}4771213 = {\cdot}1249387$.

Dividing this by 26, $\log.\ r = {\cdot}00480 = \log.\ 1{\cdot}0111$.

The first term, 3, multiplied by this number will give the 2nd term, and multiplied by its square will give the third term. To find any term, say the 11th term of the series—

By (517) 11th term $= ar^{10}$, $\therefore$ log. 11th term $= \log.\ 3 + 10 \log.\ 1{\cdot}0111 = {\cdot}52517 = \log.\ 3{\cdot}351$. This number is the 11th term of the series.

551. It is occasionally required to solve questions by the aid of logarithms which take this form :—

I. To what power must 5 be raised that it may equal 20?

Here if x = the unknown exponent, $5^x = 20$.

$\therefore x \log.\ 5 = \log.\ 20$, and $x = \frac{\log.\ 20}{\log.\ 5} = \frac{1{\cdot}30103}{{\cdot}69897} = 1{\cdot}861.$

Hence $5^{1{\cdot}861} = 20$, or 1·861 is the exponent which raises 5 to 20.

II. What would be the logarithm of 180 if the base were 12?

Here if x be the required logarithm, $12^x = 180$.

$\therefore x \log.\ 12 = \log.\ 180$, $\therefore x = \frac{\log.\ 180}{\log.\ 12} = \frac{2{\cdot}2552726}{1{\cdot}0791813} = 2{\cdot}089.$

And $12^{2{\cdot}089} = 180$, or 2·089 would be the logarithm of 180 on the base 12.

EXERCISE CXLV.

☞ Solve the following expressions by the help of logarithms :—

I. $\frac{39{\cdot}4^2 \times \sqrt[7]{4693}}{374^{\frac{4}{5}}}$; $\frac{\sqrt[11]{{\cdot}241} \times {\cdot}034^2}{67{\cdot}43^{\frac{1}{7}}}$.

II. $\left(\frac{362}{{\cdot}008}\right)^{\frac{7}{11}}$; $\frac{123 \times 74{\cdot}602}{\sqrt{7} \times {\cdot}004}$.

III. $\frac{367 \times {\cdot}073}{47}$; $\left(\frac{7}{8}\right)^{20}$.

IV. $\frac{\sqrt[5]{25 \times \sqrt[3]{432}}}{\sqrt[4]{14}}$; $\sqrt[5]{\frac{7}{3}^{4} \times \sqrt[4]{6}}$.

V. $\sqrt[4]{6 \times \sqrt{3} \times \sqrt[3]{2}}$; $\sqrt[9]{\frac{4{\cdot}5 \times {\cdot}001 \times \sqrt[3]{9}}{32^5 \times {\cdot}0005^7}}$.

VI. $\frac{4^7}{3^3} \times \frac{6^3}{4^2}$; $\frac{\sqrt{4^3 \times {\cdot}19^4}}{\sqrt[3]{6^4 \div 4^6}}$.

VII. Insert the fourth term in the proportions —

6437 : 8479 :: ·0081 : (   ); and 723·4 : ·02529 :: 3574·8 : (   )

VIII. Find the product of $\cdot007 \times 31{\cdot}5 \times {\cdot}617$.

Divide 30·04 by ·4003.

IX. $\sqrt[4]{11^8 \times {\cdot}151^4}$; $\frac{\sqrt[4]{142^{\frac{1}{12}} \times {\cdot}12^{\frac{1}{144}}}}{\sqrt[4]{19^6 \div 4^{19}}}$.

X. $\sqrt{\frac{91^6 \times 4\sqrt{2}}{7\sqrt[3]{6} \times 9\sqrt{5}}}$ $\frac{(1{\cdot}05)^7 - 1}{(1{\cdot}05)^7 \times {\cdot}05}$.

XI. $\frac{2^6 \times 25^2}{4^3 \times 10^2}$; $\sqrt[5]{363^4}$.

XII. $\frac{217{\cdot}2}{26{\cdot}53} \div \frac{{\cdot}0685}{{\cdot}0093}$; $\frac{\sqrt[3]{236}}{\sqrt[5]{7{\cdot}263}} \times \sqrt[5]{363^2}$.

XIII. $\frac{99^3 \div 6^5}{\sqrt[4]{32} \times \sqrt[3]{48}}$. XIV. $\frac{\sqrt{1040}^{1} \times \sqrt[3]{729}}{\sqrt[4]{6561}}$; $\sqrt{7788}^{\frac{1}{4}} \times \sqrt[3]{1331}$.

XV. Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

XVI. $\sqrt[5]{7\sqrt{2} \times \sqrt[3]{3}}$; $\frac{63.9 \times {\cdot}7854 \times 26}{391 \times {\cdot}08}$.

XVII. $\sqrt[4]{{\cdot}0076542}$; $\frac{\sqrt[4]{132 \times (7{\cdot}356)}}{\sqrt{(3{\cdot}25)^6}}$

APPLICATION OF LOGARITHMS—II. INTEREST.

552. *Compound Interest.*—It was stated in (381) that compound interest was to be found by working a separate sum for each of the periods mentioned in the question. The rule of Progression, however, enables us to abridge this laborious operation. For if the rate of interest on a sum of money be uniform through a series of years, the amounts at the end of each period will form a series in geometrical progression, and as the amount at the end of the first year is to the amount at the end of the second year, so is that amount to the amount at the end of the third year, &c. And if A_1, A_2, A_3, &c., represent the several amounts, it will be true that—

$A_1 : A_2 : A_3 : A_4 : \ldots\ldots A_n$ are in continued proportion.

We have first to find the constant ratio of this series.

Now if £1 be put out at interest at R per cent., at the end of the first year it will amount to $1 + \frac{R}{100}$ or $\frac{100+R}{100}$; let us call this amount A.

Then at the end of the second year the sum will amount to $A + \frac{AR}{100}$ or $\frac{100\,A + AR}{100}$ or $A \times \frac{100+R}{100}$.

But $A = \frac{100+R}{100}$; $\therefore$ amount at the end of second year $=$

$\frac{100+R}{100} \times \frac{100+R}{100} = \left(\frac{100+R}{100}\right)^2$; call this amount A''.

Then at the end of the third year, the amount will be $A'' + \frac{A''R}{100}$ $= \frac{100\,A'' + A''R}{100} = A'' \times \frac{100+R}{100}$.

But $A'' = \left(\frac{100+R}{100}\right)^2$;

$\therefore$ Amount at the end of third year $= \left(\frac{100+R}{100}\right)^3$ or $\left(1 + \frac{R}{100}\right)^3$.

And since it is evident that $\frac{100+R}{100}$, or $1 + \frac{R}{100}$, is the constant ratio of this series, it follows that—

Amount of £1 for n years at R per cent. $= \left(\frac{1 + R}{100}\right)^n$.

Amount of £P for n years at R per cent. $= P\left(1 + \frac{R}{100}\right)^n$.

RULE FOR COMPOUND INTEREST.

553. Find in a decimal form the sum to which £1 would amount at a given rate in one year.* Involve this number to the power representing the number of years, and multiply by the principal.

554. *Example I.*—Find the compound interest on £12,000 at 4 per cent. for 7 years.

By the formula, Amount $= 12000\left(1+\frac{4}{100}\right)^7 = 12000 \times 1{\cdot}04^{7}$.

$$\therefore \log. A = \log. 12000 + 7 \log. 1{\cdot}04.$$

$$\text{But } 7 \log. 1{\cdot}04 = 7\,({\cdot}0170333) = {\cdot}1192331$$

$$\log. 12000 = 4{\cdot}0791812$$

$$\log. 15791{\cdot}3 = 4{\cdot}1984143$$

Answer—£15791 6s. = Amount, and £3791 6s. = Interest.

555. *Example II.*—Find the amount of £7 at compound interest at 3 per cent. for 100 years.

Here Amount $= 7 \times 1{\cdot}03^{100}$.

$\therefore \log. A = \log. 7 + 100 \log. 1{\cdot}03$.

$$\text{But } \log. 7 = {\cdot}845098$$

$$100 \log. 1{\cdot}03 = 1{\cdot}28372$$

$$\text{Log. } 134{\cdot}53 = 2{\cdot}128818$$

Hence £134 10s. 7¼d. = Amount.

556. *Example III.*—In what time will a sum of money double itself at 5 per cent.?

Here, because Amount $= \left(1+\frac{R}{100}\right)^n$

$$\therefore \log. A = n \times \log. \left(1+\frac{R}{100}\right);$$

$$\text{And } n = \frac{\log. A.}{\log. \left(1+\frac{R}{10}\right)}$$

But if £1 be taken, $1+\frac{R}{100} = 1{\cdot}05$, and A = 2.

$$\text{And } \frac{\log. 2}{\log. 1{\cdot}05} = \frac{{\cdot}3010300}{{\cdot}0211893} = 14{\cdot}20606 \text{ years.}$$

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* If the payments be made half-yearly or otherwise the amount should be calculated for that period, and n will represent the number of half-years or intervals of payment.

EXERCISE CXLVI.

1. What is the amount of £674 15s. at 4½ per cent. compound interest for 59 years?

2. Find the amount of £603 for 17 years 6 months at 5 per cent. compound interest.

3. What would £19 19s. amount to in 91 years at 8¼ per cent. compound interest?

4. Find the compound interest on £5,600 at 6½ per cent. for 5 years.

5. Find the amount of £8,460 at compound interest at 5 per cent. for 12 years.

6. What will a capital of £12,000 amount to in 10 years at 6 per cent. per annum, the interest being paid half-yearly?

7. Find the compound interest on £450 15s. at 4½ per cent. for 12 years.

APPLICATION OF LOGARITHMS.—III. ANNUITIES.

557. An annuity is a sum of money paid at yearly intervals, and may arise from estates, from invested capital, from a pension, or any other source. Thus: the lease of an estate worth £80 a year, which will expire in 35 years, is to the owner an annuity of £80 for 35 years.

Annuities which are to last for a fixed term of years are called *Certain*, and those which are only to last during the lifetime of any particular person or persons are called *Contingent*. If an annuity becomes payable at once, it is said to be *Immediate*; if it is only to be entered upon at a certain distant period, it is called a *Reversionary* or *Deferred* Annuity.

The *amount* of an annuity means the sum of all the payments, together with the interest upon them, from the time at which the first becomes payable until the expiration of the whole term.

Now if an annuity be regularly paid at the end of each year for 20 years, interest upon the first payment accrues for 19 years; the second payment is subject to interest for 18 years, and it is only the last payment, viz., that paid at the end of the 20th year, on which no interest has to be calculated.

Suppose that interest be at 5 per cent., and that the annuity be of £1; then the last payment = £1 only; the payment of the year before is £1 + its interest for a year; that of the eleventh year requires nine years' compound interest to be added, and the several sums of money with their respective interests form a series.

20th year	19th	18th	17th	16th		1st
£1,	$1{\cdot}05$,	$1{\cdot}05^2$,	$1{\cdot}05^3$,	$1{\cdot}05^4$,	....	$1{\cdot}05^{19}$.

This series shows us each year's principal with its compound interest for the time which has to expire before the termination of the 20 years. Now the series is one in geometrical progression, the common ratio being 1·05.

Hence the total amount of the annuity is the sum of this series.

And by (524) Sum $= \dfrac{ar^n - a}{r - 1} = \dfrac{1{\cdot}05^{20} - 1}{1{\cdot}05 \quad - 1}$

$$1{\cdot}05^{20} - 1 = 2{\cdot}65329771 - 1 = 1{\cdot}65329771.$$

$$1{\cdot}65329771 \div 1{\cdot}05 - 1 = £33{\cdot}065954 = £33 \text{ 1s. } 3\tfrac{3}{4}\text{d.}$$

It follows, that if the annuity had been of £25, or of £60, or any number of pounds per annum: this sum, £33 1s. 3¾d., which represents the amount of an annuity of £1, would require to be multiplied by £25, or by £60, and that thus the amount of any such series of annual payments may always be found.

558. To find the amount of an annuity—

Find by (553) the amount of £1 at compound interest for the given time, subtract unity from this amount, multiply this difference by the annuity, and divide this product by one year's interest on £1.

Example.—What will an annuity of £48 5s. amount to in 15 years at 4 per cent.?

Here if a be the annuity the answer must be $a\left(\dfrac{r^n - 1}{r - 1}\right)$.

But $a = £48\ 5s$, $n = 15$, and $r = 1{\cdot}04$, $r - 1 = {\cdot}04$.

And the problem to solve is $\left(\dfrac{1{\cdot}04^{15} - 1}{{\cdot}04}\right) \times 48{\cdot}25$.

Now by logarithms* $1{\cdot}04^{15} = 1{\cdot}80094351$

$$\begin{array}{rr}
\text{Subtract} & 1 \\ \hline
 & {\cdot}80094351 \\
 & 48{\cdot}25 \\ \hline
 & 400471755 \\
 & 160188702 \\
 & 640754808 \\
 & 320377404 \\ \hline
{\cdot}04) & 38{\cdot}6455243575 \\ \hline
\end{array}$$

Answer—£966 2s. 9¼d. $\quad 966{\cdot}13810898$

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* These results are, in practice, more frequently obtained from Annuity Tables which have been prepared especially for this purpose, and which save the trouble of working each part of the sum by logarithms.

559. Sometimes the amount of the annuity is given, and the time, and it is required to find what the annuity itself is. In this case it is only necessary to find what would be the amount of an annuity of £1 for the given time, and divide the whole given sum by that amount.

Example.—Suppose money can be improved at 3 per cent. compound interest, how much ought I to lay by per annum in order to be worth £1000 eleven years hence?

Here, by the former rule, a sum of £1 laid by per annum at 3 per cent. for 11 years will amount to 12·807796.

Hence the required sum is $\frac{1000}{12 \cdot 8078} = 78 \cdot 077 =$ £78 1s. 6½d.

EXERCISE CXLVII.

1. Find the amount of an annuity of £325 in 9 years at £3 6s. per cent. compound interest.

2. Find the amount of an annuity of £30 6s. 4d. for 12 years at 5 per cent. compound interest.

3. The amount of an annuity of £24 at 3½ per cent. for 29 years.

4. The amount of an annuity of £283 at 6 per cent. for 23 years.

5. Find the amount of an annuity of £72 at 5½ per cent. for 17 years 6 months.

6. What sum must be put by annually for 30 years in order to amount to £500, if money is worth 3¾ per cent.?

7. What is the amount of an annuity of £48 for 31 years at £3 4s. per cent. compound interest?

8. Required the amount of an annuity of £309 for 15 years 3 months at 6 per cent. compound interest.

9. Find the amount of an annuity of £83 for 99 years at £2 17s. per cent. compound interest.

10. Find the amount of an annuity of £6,450 at 4¾ per cent for 15 years 6 months.

11. What sum must be laid by quarterly for 13 years in order to amount to £600 if money is worth 4 per cent.?

12. To what sum will an annuity of £40 2s. 6d. amount in 24 years at 2½ per cent. compound interest?

13. What is the amount of an annuity of £50 for 7 years at 5 per cent.?

14. Find the amount of an annuity of £84 10s. for 27 years at 4 per cent.

560. The PRESENT VALUE of an annuity is determined by the same considerations as apply to the deduction of discount (384). If £1 be payable a year hence, its present value is such a sum as, if improved at the current rate of interest, would amount to £1 at the end of the year. Hence—

Present value : £1 :: £1 : £1 + interest.

Suppose the rate of interest be 6 per cent.; then £1·06 due a year hence is worth £1 at this moment, and

∴ £1 due a year hence is now worth $\frac{1}{1\cdot06}$.

And because (552) £1 will be worth £$1\cdot06^2$ two years hence, at compound interest,

∴ present value of £1 due two years hence $=\frac{1}{1\cdot06^2}$.

In like manner it may be shown that—

Present value of £1 due three years hence $=\frac{1}{1\cdot06^3}$.

Present value of £1 due four years hence $=\frac{1}{1\cdot06^4}$.

Hence if a be the annuity, its present value for a number of years is represented by the geometrical series—

$$\frac{a}{1\cdot06};\ \frac{a}{1\cdot06^2};\ \frac{a}{1\cdot00^5};\ \frac{a}{1\cdot c6^4};\ \frac{a}{1\cdot06^5};\ \frac{a}{1\cdot06^6};\ \ldots\ \text{\&c.},$$

of which the common ratio is $\frac{1}{1\cdot06}$, and the Sum of the series, as determined by the rule in (524), is equal to—

$$\left(\frac{1-\frac{1}{1\cdot06^n}}{\cdot06}\right)\times a.$$

561. *Observation.*—The number representing the present value of £1, due any number of years hence, is the reciprocal of the number representing the sum to which it would amount if invested at compound interest from the present time to the expiration of the time. For the compound interest on £1 at 6 per cent. for 20 years = $1\cdot06^{20}$.

And present value of £1 at 6 per cent. 20 years hence $=\frac{1}{1\cdot06^{20}}$.

562. To find the present value of an annuity—

Divide £1 by the sum to which it would amount at compound interest for the given time. This will give the present value of £1. Subtract this sum from unity, multiply this difference by the annuity, and divide the result by the interest on £1 for one year.

Example.—Find the present value of an annuity certain, of £50, for 18 years, at 3½ per cent.

Present value of £1 due 18 years hence $= \frac{1}{1{\cdot}035^{18}} = £{\cdot}53836114$.

$$\begin{array}{rcr} 1 - {\cdot}53836114 & = & {\cdot}46163786 \\ & & \underline{50} \\ & & {\cdot}035)\underline{23{\cdot}081893} \\ & & 659{\cdot}4826 \end{array}$$

Answer—£659 9s. 7¾d.

Exercise CXLVIII.

1. If money be improvable at 5 per cent., what is the present value of an annuity of £170 for 20 years?

2. At 2½ per cent. find the present value of an annuity of £100 for 46 years.

3. At 6 per cent. what is the present value of an annuity of £25 for 34 years?

4. At 4½ per cent. what is the present value of an annuity of £234 for 16 years?

5. What sum ought to be expended in the purchase of an annuity of £150 for a person 43 years of age, supposing that the average expectation of such a life extends to the age of 58 and that money is worth only 3 per cent.?

6. How much ought to be given for the lease of a house, if 27 years of the term be unexpired, and the yearly rental be £74, calculating interest at 4 per cent.?

7. At 5 per cent. what is the present value of an annuity of £842 5s. for 12 years?

8. At 3½ per cent. what is the present value of an annuity of £216 for 8 years 6 months?

9. Find the present worth of an annuity of £643 10s. for 15 years at 3 per cent.

MISCELLANEOUS EXERCISES ON PROGRESSION AND LOGARITHMS.

1. Solve the following expressions :—

$$\sqrt[5]{\frac{10}{13}} - \sqrt[6]{\frac{5}{9}} \qquad \sqrt[6]{25} \times \sqrt[5]{347^4}$$

$$\sqrt{\frac{\sqrt[4]{32} \times \sqrt[3]{48}}{\sqrt[2]{27}}} \qquad \frac{1834^3 \times 796^7}{2584^5}$$

2. In what time will an annuity of £200 pay off a debt of £4,000 at 3 per cent. compound interest?

3. Find x in the following expressions :—

$$8^x = 100$$
$$10^x = 2$$
$$20^x = 100$$

4. In what time will a sum become ten times its original value at compound interest at 5 per cent.?

5. Insert 3 geometric means between 2 and $\frac{1}{8}$.

6. What number being raised to the fifth power will equal $(\frac{23}{417})^3$?

7. In what time will a sum of money double itself at $3\frac{1}{2}$ per cent. compound interest?

8. In what time will £20 amount to £90 at 5 per cent.?

9. In how many years will £325 amount to £395·0394 at 5 per cent. compound interest?

10. A person invests 5,000 in the 3 per cent. consols when they are at 90; what will this sum amount to in 15 years, supposing the half-yearly interest, as it becomes due, to be invested also at 3 per cent.?

11. Solve the following expressions, using logarithmic tables :—

$$\sqrt{\frac{{\cdot}006 \times 625^6}{625}}; \qquad \sqrt[123]{\frac{125^{125}}{{\cdot}125}}.$$

$$\sqrt{\frac{\left(\frac{6}{7}\right)^4 \times 7{\cdot}6^2}{726\frac{4}{5}}}; \qquad \left(\frac{2}{9}\right)^{\frac{2}{9}}.$$

$$\sqrt{546\sqrt{39} \times 845\sqrt[3]{72}}; \qquad \left(\frac{37 \times 54}{365}\right)^3 \times \left(\frac{7{\cdot}62 \times 2{\cdot}68 \times {\cdot}04}{0{\cdot}58}\right)^2$$

$$\sqrt{\left\{\frac{(9{\cdot}6425 \times 81) + (374{\cdot}5 \times 60) - (37 \times 25)}{69 \times 32}\right\}}$$

12. To what power must 50 be raised to equal 1000?

13. To what power must $\frac{23}{417}$ be raised to equal ·17577?

14. Find a fourth proportional to the 19th power of 11, the 11th power of 19, and the 17th power of 17.

15. The mean distances of Mercury and Uranus are in the proportion of ·387098 to 19·1823900, and the time of the revolution of Mercury is 87·969258 days; what is the time of revolution of Uranus, the square of the times being as the cubes of the mean distances?

16. In what time would £10 amount to £100 at 3 per cent. compound interest?

17. What will be the discount on £1,000 at 4 per cent. for two months?

18. If the discount on a promissory note of £500 1s. 3d. amounted to £32 1s. 3d., and the rate of interest 4½ per cent. compound, how long had the note to run?

19. A debt of £550, accumulating at the rate of 3 per cent., is paid off by yearly instalments of £25; when is the debt discharged?

20. What should be paid for an annuity of £25 for 14 years, allowing 4½ per cent. compound interest?

21. An annuity of £30 for 25 years is sold for £350; what interest is allowed?

22. A lease for 99 years is purchased for £100; what annual rent must be paid to allow 5½ per cent.?

23. Multiply 79368 by 27415, and divide the product by 827·145.

24. Find a fourth proportional to $726^3 : 298^4 :: \sqrt{3072} : x$.

25. A lease for 50 years is purchased for £300; what rent should be obtained to make 6 per cent. interest?

26. What is the present value of an annuity of £120 for 20 years at 6½ per cent.?

27. What would be the logarithm of 145 if the base were 16?

28. At 3½ per cent. what is the present value of an annuity of £195 for 11 years?

29. At 4½ per cent. what is the present value of an annuity of £615 for 19 years?

30. What sum of money must be annually laid by for 80 years to amount to £12,640, money being worth 5 per cent. compound interest?

31. An annuity is worth £2,500 twelve years hence, money being at 3¼ per cent.; what must be laid by annually to amount to that annuity?

32. If a person commence trade with a capital of £1,500, and each year, after paying all his expenses, increase the capital of the former year by $\frac{1}{20}$ part of itself, how much will he be worth in 50 years?

33. What did a merchant commence trade with who was worth £20,000 at the end of 20 years, if he cleared annually $\frac{1}{10}$ of his capital?

34. Insert 3 geometric means between 16 and 65536.

Questions on Progression and Logarithms.

Define Progression, Equi-different, Ascending, Descending, Differential Mean, Average. Show how the principle of Arithmetical Progression is to be referred to elementary propositions in Addition and Subtraction. How many equal sums can be found among the numbers in an arithmetical series of 7 terms?

Give formulæ for finding—(I.) The *n*th term of an equi-different series; (II.) The sum of such a series. In what sense is the word Equi-rational used? Where are the elementary truths to be found on which the laws of geometrical progression are based? What is a mean proportional? Give formulæ for determining—(I.) The *n*th term of a geometric series; (II.) The sum of such a series. How many equal products can you select from a geometric series consisting of 9 terms? Express the relation in which the last term of such a series stands to the first. Show how a recurring decimal may be regarded as an infinite geometric series. Give an instance, and show what is the common ratio.

On what main truth of arithmetic are systems of logarithms based? Define the term and give its etymology. Construct a series, giving the logarithms of 10 numbers on the base 4. What is the main purpose of logarithmic tables? How were they first employed? Explain the term *base* as applied here. Show by an example the difference between the characteristic and the mantissa. How many inferences can you deduce from the fact that log. 23 = 1·3617278? How is the characteristic determined if it is not given in the tables? Show the reason of this. Describe the mode of calculating differences, and give an example.

What is the difference between compound interest and an annuity? Give formulæ adapting these rules to logarithmic computation. What is generally the common ratio in calculating compound interest in annuities?

State in words the truths embodied in the following formulæ:—

1. $n\text{th term} = a + (n - 1)d;\qquad S = \frac{(a + z)n}{2} = (2a + (n - 1)d)\frac{n}{2}.$

2. $n\text{th term} = ar^{n-1};\qquad a : \sqrt{ab} : b.$

3. $r = \sqrt[n-1]{\frac{z}{a}};\qquad S = \frac{ar^n - a}{r - 1} = \frac{(r^n - 1)a}{r - 1}.$

4. $a^m \times a^n = a^{m+n};\qquad \frac{a^m}{a^n} = a^{m-n}.$

5. $\text{Log. } abc = \text{log. } a + \text{log. } b + \text{log. } c$

6. $\text{Log. } a^n = n \times \text{log. } a;\qquad \text{log. } \sqrt[n]{a} = \frac{\text{log. } a}{n}.$

APPENDIX.—A.

MONEY, WEIGHTS, AND MEASURES.

I. THE UNIT, OR STANDARD OF MEASUREMENTS.

IT was shown in (4) that before Arithmetic can be applied to the measurement of any concrete magnitude, the *unit* of that magnitude must be clearly defined and understood. But such quantities as *length*, *solidity*, *weight*, *time*, and *value*, being capable of continuous increase, offer us no units which are clearly distinguishable, and by means of which they can be readily compared with one another. Men are compelled, therefore, to select arbitrary standards; and it is not surprising that almost every nation has its own peculiar method of measuring, and its own set of tables. On this account there is great difficulty in comparing ancient or foreign measures with our own.

Of all the magnitudes just mentioned, time is the easiest of measurement; * because the period of the earth's revolution round its axis, or one day, and the period of the earth's revolution round the sun, or one year, are portions of time which, as far as we know, never vary. They form the natural standards by which all nations alike measure their time. The day is the principal unit of time in all countries of the world, and all other periods are either aliquot parts or multiples of a day. Nations may differ as to the subdivisions of a day, but they cannot differ as to the day itself. Hence there is no difficulty in comparing foreign methods of computing time with our own; because nature has provided all men alike with two fixed unalterable units to serve as the basis of their calculations.

It is to be desired that some natural standards could be found, in like manner, to which lengths, values, bulks, weights, and other concrete magnitudes could be universally referred. But it is very difficult to find such standards in nature, and all the inconsistencies and difficulties of tables arise out of the fact that the standards in common use have been arbitrarily chosen, not on any system, but chiefly by accident. For instance, an object in nature which is always precisely of the same length, and which will therefore serve as the unit of linear measurement, is not easy to find. If it were found, it would be possible to form from it a unit of surface, and so to found square measure upon long measure. Then, from the square unit of surface, we might obtain a cubic unit, or fixed standard of bulk, and this would serve also as a measure of capacity. Then, if some substance were chosen whose weight never varied, enough of this substance to fill a certain measure might be taken to serve as the unit of weight. In this way, superficial measure, measures of capacity (including ale, beer, wine, and dry measures), and measures of weight, could all be connected with the unit of length, and one fixed standard of length would suffice to determine all these measurements in a permanent way. We shall see that our own and most other systems rest to some extent on this principle, and that the fundamental difficulty is to determine the *linear* unit.

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* It may be noted that, if it were not for the circumstance mentioned in the text, time would be of all magnitudes the hardest to measure. For in comparing magnitudes of any other kind, equality or inequality can be determined by bringing them together, and observing whether they coincide or not. This is manifestly impossible in the case of two periods of time; and but for the uniformity of the laws of nature we should find it impossible to verify our calculations respecting the duration of time.

II. MEASURES OF VALUE—MONEY.

Measures of value are of all others the least easy to compare and to understand. For there is nothing in nature which is always exactly of the same worth to men, or for which men will always give precisely the same amount of labour or of time. Hence there are hardly any two countries which use the same unit of value; and the unit itself varies much in any given country at different times. Gold and silver have been chosen in most civilized countries as the standards to which all other values are reduced. The reasons for this selection are—1. Because, compared with other commodities, their value fluctuates very little. 2. Because they are hard and durable materials. 3. Because they are capable of very accurate and minute subdivision. 4. Because, compared with other things of equal worth, they occupy very little space, and can easily be removed from place to place. Nevertheless gold and silver, like other valuable things, rise and fall in price according to the quantity in the market; and even if they did not, we could not always measure the wealth of other times or other nations by a money standard. For in order to tell the worth of a given sum of money, we must know what amount of comforts or commodities can be purchased with it; and this varies very much under different circumstances.

The value of an ounce of gold * is £3 17s. 10½d in this country, and this price has of late undergone very little fluctuation. The value of silver varies more, and is generally about 5s. per ounce.† It would therefore seem that the true way of comparing the value of foreign coinage with our own is to ascertain the fineness and weight of the metal of which it is made, and calculate its worth by this standard. But in fact the relative value of coinage in different countries is determined also by the state of trade between them. For when two countries have commercial intercourse, money does not actually pass from one to another, but bills are drawn (388) and are negotiated among the traders instead Thus, if a French merchant has an English creditor, and an English merchant has a French creditor, it will not be necessary that each should ship gold or silver to pay his debts, if the former pays to his neighbour the sum due to the Englishman, and the latter pays to that Englishman the money due to his French correspondent. If all the Englishmen who owe money in France accept bills for the amounts, and all the Frenchmen who are indebted to persons in England do the same, these documents can be negotiated in the respective money markets, and no specie need be transmitted. Now if there be such an equilibrium in the trade of the two countries as that there is just the same sum owing by the whole body of English merchants that the whole body of French merchants owe to England, there will be as many bills drawn in England and payable in France, as are drawn in France and payable in England. There will be just the same number of English debtors wishing to purchase bills upon France, as of English creditors wishing to dispose of them. The exchange is under such circumstances said to be *at par.*

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* This is not to be understood as perfectly pure gold; $\frac{1}{12}$ of it is supposed to be alloy, so that it would be more accurate to state that $\frac{11}{12}$ of an ounce, or 440 grains, represents the amount of pure gold in £3 17s. 10½d.

† A very serious fall in the price of silver took place in 1873, owing partly to the discovery of new silver mines in Nevada, and partly to the introduction of a gold standard into Germany, and the consequent diminution in the demand for silver coin. The price of bar silver per oz. fell to 59½d., and during three subsequent years continued to diminish, until in June, 1876, it ranged from 52d. to 50d. The price of silver has, however, since risen.

Suppose, however, that France is exporting more to England than we are sending to her in return. The sum of money owing by English merchants to France will be greater than that in which France is indebted to us. Hence in the French money market the supply of English bills payable to French merchants will be greater than the demand, and such bills will be sold at a discount; while in England French bills will be at a premium, or will sell for rather more than they are worth. As in this case a balance of money is due to France, the exchange is said to be in favour of that country.

For example, an English sovereign exchanges for 25 francs 22 centimes when the exchange is at par. But if the exchange be in favour of France, English money is temporarily at a discount, and will perhaps be reckoned equivalent to 25 francs 10 cents in business transactions. On the other hand, when our money bears a premium, the course of exchange being in favour of England, a sovereign will bear a higher value, and may be reckoned at 25 francs 30 centimes, or even 25 francs 50 centimes.

The state of the market at any time is called the *course of exchange.* It is necessarily affected by other circumstances,—the credit of the persons on whom the bills are drawn, the mint value of the specie in different countries, &c. But there is a limit to the fluctuations in the course of exchange. For though the transmission of specie involves inconvenience and risk, yet if the exchange were unusually to the advantage of France, a merchant in England would rather incur the expense of sending money to that country than purchase bills at a very high premium, and therefore at a loss. In France the standard of value is a Franc, or 5 grammes of sterling silver. Its value is equivalent to 9·516d. of English money.

ENGLISH MONEY.

A FARTHING ($\frac{1}{4}$d.)	= $\cdot 001041\dot{6}$ of £1.
A PENNY (1d.)	= 4 farthings = £ 00416.
A SHILLING (1s.)	= 12 pence = 48 farthings = £·05.
A SOVEREIGN or Pound (£1)	= 20 shillings = 240 pence = 960 farthings.

A sovereign weighs 5 dwt. $3\frac{171}{623}$ grains, or 123·274 grains of gold.

A shilling weighs 3 dwt. $15\frac{3}{11}$ grains, or 5·6552 grains of silver.

A penny weighs 145·833 grains of bronze, which is a mixed metal composed of ·95 copper, ·04 tin, and ·01 zinc.

The intrinsic value of the gold coinage is the same as its nominal value. But the silver and bronze pieces are merely tokens used for convenience; and are not of the full value which they represent. Hence a creditor can when money is owing claim to be paid in gold. Silver, except for sums under forty shillings, is not a legal tender.

L. S. D. are initials of the Latin words Libri, Solidi, Denarii, which are the plural forms of the names of three Roman coins. They have been applied respectively to Pounds, Shillings, and Pence, although these words do not represent exactly the same values. Until the alteration in the French coinage, the same three words were employed in the form of *livres*, *sous*, and *deniers*.

The following is a list of various coins—now, with the exception of the crown, obsolete—which have been in use in this country in past times.

A Moidore	= 27s.	A Guinea	= 21s. 0d.	A Noble	= 6s. 8d.
A Jacobus	= 25s.	A Mark	= 13s. 4d.	A Crown	= 5s. 0d.
A Carolus	= 23s.	An Angel	= 10s. 0d.	A Groat	= 4d.

TABLE OF FOREIGN MONEY.

	Sterling Money.	Decimal of £1.	Value of £1 sterling at par.
	s. *d.*		
(*a*) FRANCE, ALGERIA, BELGIUM, and SWITZERLAND.—1 *Franc* = 10 decimes = 100 centimes	0 $9\frac{1}{2}$	·039	25 frs. 22 centimes
(*a*) ITALY.—1 *Lira* or Franc = 100 centimes	0 $9\frac{1}{2}$	·039	25 lira 22 centimes
(*a*) SPAIN.—1 *Peseta* or Franc = 100 centimes	0 $9\frac{1}{2}$	·039	25 pesetas 22 cents.
AUSTRIA.—1 *Florin* = 100 new kreuzers	1 $11\frac{3}{4}$	·099	10 fls. 28 kreuzers

(*a*) Since 1865, France, Belgium, Switzerland, and Italy have adopted the same unit of value. It is a silver standard, though gold pieces are coined in each country. These states constitute what is called the Latin Monetary Union.

	Sterling Money.		Decimal of £1.	Value of £1 sterling at par.
	s.	*d.*		
(*b*) GERMANY.—1 *Mark* = 100 pfennings	1	0	·05	20 marks
(*c*) RUSSIA.—1 *Rouble* = 100 copecks	3	2	·158	6 roubles 40 copecks
HOLLAND.—1 *Florin* or Guilder = 100 cents	1	8	·083	12 fls. 9 cents.
PORTUGAL, and BRAZIL.—1 *Milrea* = 100 reas	4	5½	·223	4 milreas 500 reas
DENMARK, SWEDEN, and NORWAY.—1 *Kronor* = 100 öre	1	1½	·056	18 kronors
TURKEY and EGYPT.—1 *Piastre* = 40 paras	2	1	·009	110 piastres
GREECE.—1 *Drachma* = 100 centimes or leptu	0	9	·0037	28 drachm. 50 leptu
(*d*) UNITED STATES of AMERICA and MEXICO.—1 *Dollar* = 100 cents	4	2	·208	4 dollars 84 cents
(*e*) EMPIRE of INDIA.—1 *Rupee* = 16 annas	2	0	·1	10 rupees
CHINA.—1 *Tael* = 10 mace = 100 candarines	5	10	·29	3 taels 7 mace

(*b*) Up to 1875, the chief coin in use in Prussia was the *thaler*, equal to about three English shillings, and other German states used coins of various values. In 1875, the new system of coinage was legally established for all the German states included in the Zollverein, among which the most important are Prussia, Saxony, Hanover, Hesse, Bavaria, Wurtemburg, Baden, Saxe-Coburg, Nassau, and the cities of Frankfort and Hamburg. Though the *mark* is the unit of account, the 20 mark piece, or gold coin is the chief standard of value.

(*c*) An Imperial is a gold coin worth ten roubles.

(*d*) The American Eagle is a gold coin of ten dollars.

(*e*) A lac of Rupees equals 100,000; and when silver is at its normal value is worth 10,000 sterling.

III.—MEASURES OF LENGTH.

In early and rude calculations the parts of the human body formed the principal standards of length. The terms foot, hand, palm, span, and pace, indicate the origin of our ordinary measures, and words equivalent to these are found in the languages of the Greeks and Romans. Among the Romans the *pes* or foot was the principal unit, and was equal to 11·16[*] modern English inches; the *uncia*, or inch, being $\frac{1}{12}$ or the breadth of the thumb, and the *digitus*, or finger breadth, being $\frac{1}{16}$ of a foot. The breadth of the four fingers (measured across the joints) gave the *palmus*, or hand, which was ¼ of the foot. The *cubitus* was the length from the elbow to the tips of the fingers, or 1 foot and a half. 2½ feet were one *gradus* or step; 2 steps, or 5 feet, one *passus* or pace; 1,000 paces one *milliare* or mile.[†] Of course it became necessary in very early times to fix an average standard for each of these lengths; and specimens of the legal lengths were probably kept in public buildings for reference and appeal.

The measurement in use in this country during the Middle Ages was founded on that of the Romans, but as the original standards were lost it became necessary to fix upon some others; for this purpose grains of barley were employed. The tradition is that at first four grains placed side by side were equivalent to the *digitus* or lowest measure.[‡] But by a statute of Edward II. it was enacted that the length of three barleycorns, round and dry, taken from the middle of the ear, should make the legal inch, 12 inches one foot, 3 feet one yard, &c. It is commonly said that the legal measure of a yard was taken from the arm of Henry I., [§] but this is very doubtful. Standards were made for reference, and ordered to be kept in all large towns: these, however, were necessarily exposed to accident; and in fact the standard yard was destroyed in the fire at the Houses of Parliament, 1834. In ancient books of arithmetic also it was customary to illustrate the tables by printed lines or diagrams representing the various lengths; but this kind of standard is unsatisfactory, as paper is liable to shrink, and books are easily defaced or destroyed.

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[*] The Greek πους (*pous*), or foot, was 12·14 English inches, or rather more than an English foot. This is ascertained by the measurement of some buildings still existing, whose dimensions are given by ancient writers. Yet the average length of a human foot is actually about 10·3 inches.

[†] The assumption on which these measures are based, and which is on the whole an accurate one, is that in a man of average proportions the breadth of the palm is $\frac{1}{24}$ of his height; the length of the foot one-sixth, and that of the cubit one-fourth.

[‡] If this were true, 64 grains placed side by side would equal an English foot; but on putting this to a simple test it may be found that 64 grains of even well-grown barley give little more than ⅝ of a foot.

[§] William of Malmesbury states that Henry I. commanded that the *ulna*, or ancient ell, which answers to the modern yard, should be made of the length of his own arm. But this is the only authority (and not a very trustworthy one) for the fact.

For all ordinary commercial purposes such standards as these are sufficiently accurate, the average length of barleycorns being probably uniform in successive years. But for scientific purposes it is desirable that the standard should be determined with mathematical exactness. It is for such purposes only that so many laborious investigations have been made with regard to the unit of length.

The object, therefore, in regard to these measures was to discover some one fact or object in nature which would furnish us with an unalterable standard of length. As physical science became more studied, the attention of learned men was directed to this subject. It was found that in a given latitude the pendulum which marked seconds was always of exactly the same length,* and as the length of the seconds pendulum in London is 39·1393 inches, a ready means was discovered for correcting any future deviations from the true standard length by referring it to the standard of time.†

It has now been shown—1st. That of all the magnitudes we wish to measure, *time* is that of which nature has most distinctly fixed the standard. 2ndly. That a method has been ascertained of determining the measure of length by that of time. 3rdly. That from a fixed length it is possible to deduce fixed measures of surface, of capacity, and of weight.

In the year 1826 an Act of Parliament was passed which settled the national usage in regard to weights and measures, abolished many that were cumbrous and inconvenient, and explained the connection of the several tables with that called Long Measure, and of all of them with time. The following is the table in ordinary use :—

LONG MEASURE.

1 Inch‡	= $\frac{1}{39{\cdot}13}$ of seconds pendulum in the latitude of London			
1 Foot	= 12 inches			
1 Yard	= 3 feet	= 36 inches	= ·914883 French metres	
1 Pole	= 5½ yards	= 16½ feet	= 198 inches	
1 Chain	= 4 poles	= 100 links	= 792 inches	
1 Furlong	= 40 poles	= 10 chains	= 220 yards	= 660 feet
1 Mile	= 8 furlongs	= 1760 yds.	= 5280 ft.	= 63360 in.

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* The Royal Commission of 1843, appointed to investigate this matter, reported that although the reference of the standard of length to the seconds pendulum was of great importance, on the whole they judged it better not to depend on such tests, but to cause a number of exact copies of a standard yard to be made and deposited in secure places throughout the country. Gun-metal, composed of copper, tin, and zinc, in the proportion of 16, 2½, and 1, was recommended as the best material, being hard, elastic, and less affected by change of temperature than others.

† Not that a second is a period of time which nature measures for us, but because it is an aliquot part of the natural period called a day.

‡ We have no names for the subdivision of an inch, which may be eighths, twelfths, or tenths ; barleycorns are entirely out of use.

CLOTH MEASURE.

1 Nail = $2\frac{1}{4}$ inches
4 Nails = 1 quarter
4 Quarters = 1 yard = 16 nails = 36 inches.

An English ell is 5 quarters, a French ell 6 quarters, and a Flemish ell 3 quarters. This measure is now nearly obsolete.

MISCELLANEOUS OR OBSOLETE MEASURES OF LENGTH.

Hand (for measuring horses) = 4 inches; a *Span* = 9 inches; a *Cubit* = 18 inches; a *Fathom* = 6 feet; a *Chain* = 22 yards; a *Degree* = 69·1 miles; a *Geographical Mile* or *Knot** = $\frac{1}{60}$ of a degree; a Greek *Stadium* = ·1149 English mile; a *Parasang* = 30 stadia, or $3\frac{1}{4}$ English miles; a *Log Line*, used by sailors, = 48 feet; a *Piece* of calico = 28 yards; a *Piece* of Irish linen = 25 yards; a *Piece* of muslin = 10 yards.

THE METRIC SYSTEM—MEASURES OF LENGTH.

The entire system of weighing and measuring in France was remodelled in 1791 with a view to the reduction of all tables to a decimal form. The plan adopted was to fix on one definite length, one surface, one cube, and one weight (the last three being founded on the first), and to employ the same syllables in each table for the measures and multiples of these units. To express the various decimal multipliers, terms were taken from the Greek language, while for the decimal divisors the syllables were chosen from the Latin; *e.g.*,—

GREEK.		LATIN.	
Kilo (from *chilia*)	= 1,000	Deci (from *decem*)	= $\frac{1}{10}$
Hecto (from *hekaton*)	= 100	Centi (from (*centum*)	= $\frac{1}{100}$
Deca (from *deka*)	= 10	Milli (from *mille*)	= $\frac{1}{1000}$

FRENCH MEASURE OF LENGTH.

The unit of length chosen is the ten millionth part of the distance from the equator to the pole; it is called a *Metre*, and is equal to 39·371 English inches, which is nearly the same length as the seconds pendulum. The table is as follows:—

Kilo-metre	= 1000 metres =	39371 English inches
Hecto-metre	= 100 metres =	3937·1 ,,
Deca-metre	= 10 metres =	393·71 ,,
Metre, principal unit	=	39·371 ,,
Decimetre	= $\frac{1}{10}$ of a metre =	3·9371 ,,
Centimetre	= $\frac{1}{100}$ of a metre =	·39371 ,,
Millimetre	= $\frac{1}{1000}$ of a metre =	·039371 ,,

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* A *knot*, or nautical mile—one 21,600th part of the earth's circumference at the equator—contains 2,028 yards, and is to a statute mile as 2,028 is to 1,760, or as 59 to 60.

The measure most frequently quoted in France for long distances is the kilometre. It measures 1093 yards 1 foot, and is about 5 furlongs. Hence 8 kilometres are equal to about 5 English miles.

IV. MEASURES OF AREA OR SURFACE.

These measures are obviously dependent on the preceding. A square surface having a given unit of length for its base is the best measure of area.

Land or Square Measure.

1 Square Inch = a square surface having a linear inch for each of its [sides
1 Square Foot = 144 sq. inches
1 Square Yard = 9 sq. feet = 1296 sq. inches
1 Square Pole = 30¼ sq. yds.
1 Square Rood = 40 sq. poles = 1210 sq. yds.
1 Acre = 4 sq. roods = 4840 sq. yds. = 100000 sq. links
1 Square Mile = 640 acres

The acre is the principal unit of land measurement. It contains 10 square chains, and 40·4671 French ares, or ·404671 of a Hectare.

Miscellaneous or Obsolete Measures of Surface.

A *Hide* of land = 100 acres
A *Yard* of land = 30 acres
A *Square of Flooring* = 100 sq. ft.
1 *Rod of Brickwork* = 272¼ sq. ft.

1 Quire of paper = 24 sheets
1 Ream = 20 quires
1 Printer's Ream = 21½ quires

French Measure.

To form the unit of surface, the following is the method employed. A *decametre*, or line 10 metres long, is taken, and on this is described a square. This square is the French unit of surface. It is called an *Are*. As ten such surfaces cannot of themselves form a square, the words *Decare* and *Deciare* are not in use. The table of superficial measure is as follows:—

Hectare = 11960·46
Are, principal unit = 119 6046 square yards English.
Centiare = 1·196046

Large surfaces are generally spoken of in France as *Hectares*, each of which is about 2½ acres English.

V. MEASURES OF SOLIDITY OR CAPACITY.

These may easily be deduced from those of surface and of length, for a solid in a cubical shape. and having a given unit of length for one of its edges, is the only standard we need, either to measure the bulk of solids or the capacity of vessels.

Nevertheless the old English measures were not formed in this way, but by a less direct method. A statute of Henry III. (1266) enacts that 32 grains of wheat well dried shall be the legal weight of a silver penny, and shall be called a pennyweight; twenty such pennyweights shall make an ounce, 12 ounces a pound, and 8 pounds of wheat thus counted and weighed should fill a gallon. Thus the gallon became the standard unit of capacity, and was derived from weight, and not immediately from length.

But this is an unsatisfactory and unscientific mode of obtaining the unit, and by the Act of 1826 it was ordered that the gallon should contain exactly 277·274 cubic inches. Before this Act the gallon used to measure corn was less than that for wine, as 268·6 to 282. On this account it was necessary that the *shape* of the measuring vessel for dry goods should be fixed by Act of Parliament, and it was further directed that the articles thus measured were to be heaped above the rim of the vessel in the form of a cone, to one-third of the height of the measure. Now that heaped measure is abolished, and the exact number of cubic inches is fixed, it is unnecessary to insist upon any particular form of the vessel.

The *Gallon* is the standard measure of capacity, both for dry goods (corn, &c.) and for liquids. Special names are employed in Wine, Ale, and Beer Measures, but these are rather names of casks than standard measures, for by the Act of Parliament the contents are always to be gauged in gallons. The imperial gallon contains about 10lbs. avoirdupois of pure water.

SOLID MEASURE.

1 cubic foot = 1728 cubic inches = ·02831 French cubic metres.
1 cubic yard = 27 cubic feet = ·764513 French cubic metres.

IMPERIAL MEASURE.

A Gill = ¼ of a pint = 8·665 cubic inches
A Pint = = ⅛ of a gallon
A Quart* = 2 pints = ¼ of a gallon
A *Gallon*† = = 277·274 cubic inches = 4·5435 litres
A Peck = 2 gallons = 8 quarts
A Bushel = 4 pecks = 8 gallons
A Quarter = 8 bushels = 64 gallons
A Load = 5 quarters = 320 gallons

For medical purposes 1 pint is divided into 20 fluid ounces, and fluid ounce into 8 fluid drachms.

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* *Quart*, from *quartus*, = one-fourth.

WINE MEASURE.

1 Anker	=	10	gallons
1 Runlet	=	18	,,
1 Tierce	=	42	,,
1 Puncheon	=	84	,,
1 Hogshead	=	63	,,
1 Pipe	=	126	,,
1 Tun	=	252	,,

ALE AND BEER MEASURE.

1 Firkin	=	9	gallons
1 Kilderkin	=	18	,,
1 Barrel	=	36	,,
1 Hogshead	=	54	,,
1 Butt	=	108	,,
1 Tun	=	216	,,

MISCELLANEOUS MEASURES OF SOLIDITY AND CAPACITY.

1 Sack = 3 bushels; 1 Chaldron = 36 bushels; 1 *Load* or *Ton of hewn timber* = 50 cubic feet; 1 *Load of rough timber* = 40 cubic feet; 1 *Ton* (in shipping) = 42 cubic feet; 1 *Ton* of marble = 12 cubic feet; 1 *Ton* of Portland stone = 16 cubic feet; 1 *Ton* of Bath stone = 20 cubic feet.

FRENCH MEASURES.

I. Solidity.—The unity of bulk for measuring large masses of timber, &c, is thus formed. Take a *Metre* and form a cube having this length for each of its edges. This mass is called a *Stere.*

Decastere	= 10 steres	=	353·17	cubic feet
Stere, principal unit			35·317	,,
Decistere	= $\frac{1}{10}$ of a stere	=	3·5317	,,

II. Capacity.—The unit employed for the measurement of corn and dry goods, as well as liquids, is the cube of the *Decimetre*, and is called a *Litre.* This is nearly equal to two English pints, and contains 61·028 cubic inches. The following is the table :—

Hectolitre	=	6102·8	cubic inches	
Decalitre	=	610·28	,,	
Litre, principal unit	=	61·028	,,	= 1·76 English pints.
Decilitre	=	6·1028	,,	
Centilitre	=	·61028	,,	

A Hectolitre contains about 2¾ English bushels.

VI. MEASURES OF WEIGHT.

It has been seen that at one time grains of barley, carefully numbered or measured, formed the standard of weight. From this it would appear that 7680, or 32 × 20 × 12, was the number of grains in an old pound. But as this mode of weighing is liable to variation, a method was sought by which the unit of weight might be referred to that of capacity. Now water is universally accessible, and at a given temperature, and under a given barometric pressure, it never varies in specific gravity. A cubic foot, or 1728 cubic inches of water, at a temperature of 62 degrees, and when the barometer is at 30 inches, is found to weigh nearly 1000* ounces avoirdupois, or 62·32106 pounds. Hence the unit of capacity may be made to furnish a measure of weight. But the experiments necessary for verifying the accuracy of this result are of extreme nicety and delicacy; and it was recommended by the commissioners who reported to Parliament on this subject in 1843, that standards of metal should be made and kept by the nation, in preference to deducing the unit of weight from that of length.

The weight now called Troy weight is undoubtedly of more ancient use than that known as Avoirdupois. The troy pound consists of 5760 grains (for though 32 grains originally made a pennyweight, 24 was the number even before the time of Henry VII.). This king is said to have fixed the troy pound at its present weight, and to have increased the old Saxon pound by three quarters of an ounce. By the statute of 1826 the use of this weight is limited to gold, silver, platinum, diamonds, precious stones, and such drugs as are sold by retail. Until the time of Henry VIII. gold and silver were weighed by the Tower pound, which was 11¼ ounces.

There was a weight in use in business up to this period called the merchant's pound, of 15 ounces troy. This is plausibly conjectured to be the origin of the modern avoirdupois pound, which is 14·6 ounces troy. There is good reason to believe that when the word avoirdupois first came in use it was applied to this greater pound. The present standard of the avoirdupois pound was fixed in the reign of Elizabeth. It contains 7000 troy grains. It therefore is to a troy pound as 7000 is to 5760.

The great or merchant's pound consisted of 7680 grains, and is the same as the old pound just described, which had 32 grains to a pennyweight; of these grains 20 made 1 scruple, 3 scruples 1 drachm, 8 drachms an ounce, 16 ounces 1 pound. It appears that long after the settlement of the avoirdupois standard medicines continued to be dispensed by this ancient subdivision of the pound. Since 1864, however, the use of scruples and drachms in apothecaries' weight has been abandoned, and drugs are now sold by the avoirdupois pound, which for purposes of more minute subdivision is held to consist of 7000 grains. It will be seen that—

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* The exact number of ounces is 997·14.

$$\text{A troy oz.} : \text{an avoirdupois oz.} :: \frac{5760}{12} : \frac{7000}{16} :: 192 : 170.$$

TROY WEIGHT.

1 Pennyweight = 24 grains = 1.555 French grammes
1 Ounce = 20 dwt. = 480 grains
1 Pound = 12 oz. = 240 dwt. = 5760 grains = 373·24 grammes

APOTHECARIES' WEIGHT.

1 Ounce (℥) = 473½ grains
1 Pound = 16 ounces = 7000 grains.

AVOIRDUPOIS WEIGHT.

1 Dram = one sixteenth of an ounce.
1 Ounce = 16 drams = the weight of $\frac{1}{997\cdot14}$ of a gallon of distilled water
1 Pound = 16 oz. = 256 drams = 7000 grains = 453·6 French [grammes.
1 Quarter = 28 lbs. = 448 ounces = 7168 drams
1 Cwt. = 4 qrs. = 112 lbs. = 1792 oz. = 28672 drs.
1 Ton = 20 cwt. = 80 qrs. = 2240 lbs. = 35840 oz.

MISCELLANEOUS AND OBSOLETE MEASURES OF WEIGHT.

A Stone = 14 lbs. (avoirdupois); a Stone of meat = 8 lbs.; a Sack of coals = 2 cwt.; a Truss of straw = 36 lbs.; a Truss of hay = 6 lbs.; a Load = 36 trusses; a Pack of wool = 240 lbs.; 1 Firkin of butter = 56 lbs.; 1 Fother of lead = 19½ cwt.; a Stone of glass = 5 lbs.; a Pocket of hops = 112 lbs.; a Firkin of soap = 64 lbs.; a Gallon of salt = 7 lbs.; a Bag of rice = 168 lbs.; a Chest of tea = 84 lbs.; a Quintal of fish = 112 lbs; a Barrel of anchovies = 30 lbs.; a Barrel of flour = 196 lbs.; a Gallon of oil = 9 lbs.

FRENCH MEASURES OF WEIGHT.

The unit of weight is thus derived. A *Centimetre* or $\frac{1}{100}$ of a metre is taken, and a cubical vessel which has this length for one of its edges is filled with distilled water. The weight of the quantity of water thus measured is the standard unit. It is called a *Gramme*.

Kilogramme = 15444·0234 grains
Hectogramme = 1544·4023 ,,
Decagramme = 154·4402 ,,
Gramme, principal unit = 15·4440 ,,
Decigramme = 1·5444 ,,

The kilogramme is the weight most frequently employed, and is about 2 lbs. 3 oz. 4 drs. Avoirdupois. An English pound is equal to 45·36 decigrammes.

VII. MEASURES OF TIME.

We have seen that a day is the universal unit employed for measuring time. However varied the mode of reckoning smaller periods may be among different nations, there can be no difficulty in comparing them, as they are all aliquot parts of a day. The only difficulty attending the construction of these tables is, that there are longer periods of time indicated by nature. such as that of the revolution of the moon round the earth, and of the revolution of the earth round the sun; and these two periods, *a month and a year, are incommensurable with a day and with each other*. We can neither express a year nor a lunar month exactly by days and fractions of a day. The solar year is 365·2422414 days nearly, and this fraction cannot be expressed by any exact number of hours, minutes, or even seconds. Again: the period of one lunation, or the time between two new moons, is 29·5305887 days; and this fraction is likewise incapable of being accurately expressed in minutes and seconds.

Though the determination of a day is very simple, it required very careful observation to register the exact length of the solar year. Hence among semi-barbarous nations it was commonly the case that a number of days either too great or too small was fixed upon to represent the year. Thus, the Egyptians reckoned the year of 360 days. Numa, the second king of Rome, formed the year of 12 lunar months (of 29 and 30 days alternately), or 354 days. To make this arrangement agree with the actual solar year, an intercalary or extra month was introduced every two years. But as the length of this extra month, and even the periods for its occurrence, were left to the discretion of the pontiffs, great irregularities occurred. In the later time of the republic the additional month was not inserted so often as was necessary, and the average nominal year of the Romans was considerably less than the solar year. The effect of this was to make the artificial computation in advance of the real time; and when the error had accumulated to rather more than two months, the Romans were, in fact, calling that the beginning of June which should have been the end of March.

To remedy this evil, Julius Cæsar, in the year 47 B.C., caused the year to consist of 445 days, thus adding 67 days to the ordinary calculations. This exceptional year has been sometimes called the year of confusion, and the effect of the provision thus made was to bring the nominal year into harmony with the natural one. At the same time, with the aid of some Egyptian astronomers, he ascertained pretty nearly the exact length of the solar year. He took it to consist of 365¼ days, and enacted that every fourth year there should be an additional day to make up for the four quarters. As this additional day was added by causing the sixth of the calends of March (23rd of February) to be repeated twice, the year in which the day was added was called bissextilis, (*bis* = twice, *sextilis* = sixth), or the year in which there were two sixths of March.

The Julian calendar thus fixed was adopted throughout Europe. But it was based on the supposition that the year contained 365 days

6 hours (365·25 days), whereas the true year only contains 365 days 5 hours 48 minutes 49·7 seconds (365·2422414). Thus an error of little more than 11 minutes per annum was caused. This error, it should be observed, was an error of the opposite kind to that which the Romans had made; for whereas, by making their artificial year too short, their calculations were in advance of the real year; the Julian calendar, on the other hand, by making each artificial year 11 minutes longer than the real one, allowed the true year to be in advance of the ordinary calculation.

In the sixteenth century it was observed that the error thus caused had accumulated to ten days. Men were calling that the 10th of March which ought to have been called the 20th. Pope Gregory XIII., in 1582, proposed to correct this by omitting ten days from the nominal year; and with a view to prevent the recurrence of the mistake, he enacted that the extra day should be omitted in every 100th year, except the 400th. Thus of the extra days fixed by the Julian calendar to be inserted, one in every fourth year, the Gregorian correction omits three in every 400 years. So that the years 1700, 1800, and 1900, were not to be leap years, but 2000, 2400, 2800, are to have a 29th of February as usual. By this arrangement, though it is not a perfect one, it is calculated that the error in computation will not amount to one day in 6000 years.*

It is important to remember that this arrangement of time, called the *new style*, did not take effect in this country until 1751, when the error had amounted to eleven days. All earlier dates, therefore, which occur in history, or in old documents, require to be corrected by the addition of 11 days.

TIME TABLE.

1 Minute = 60 seconds
1 Hour = 60 minutes
1 Day = 24 hours
1 Week = 7 days.

The year is rather irregularly divided into 12 parts, averaging 30·416 days each. Each of these is called a *Calendar* month: 28 days, or 4 weeks, are called a *Lunar* month.

MISCELLANEOUS MEASURES OF TIME.

1 Lustrum = 5 years; 1 Century = 100 years; 1 Lunar Cycle = 19 years; 1 Solar Cycle = 28 years.

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* It is interesting to observe how very reluctantly this arrangement was adopted by Protestant countries. The change took effect in France, Spain and Portugal, and Italy, immediately on its promulgation by the Pope in 1582. The new style was legally established throughout the Netherlands in the same year; but several of the provinces refused to adopt it, and in Utrecht and Guelders it was not in use until 1700. Throughout Germany and Switzerland the Catholics received it in 1584, the Protestants not until 1699. In Sweden the new calendar commenced in 1753. In our country the change was resisted until 1751, and was very unpopular. To this day Russia and the countries in connexion with the Greek Church adhere to the old style.

Appendix II.—Various Methods of Notation.

The method of notation which represents large numbers by varying the position of a few figures would apply equally well if any other number than TEN were chosen as the basis.

It has been shown, that because we have taken ten for the base of our notation—I. We need distinct significant characters for the first *nine* numbers only, and express all higher numbers by varying their positions; and II. Every higher number than 9 has one part which is considered as a product, having for one of its factors *ten*, or some power of ten. Thus—

$$22222 = 2\,(10^4) + 2\,(10^3) + 2\,(10^2) + 2\,(10) + 2.$$

Now it follows, from similar considerations, that if 4 had been chosen for the base of our system we should only have needed *three* significant figures; 10 would have meant *four*, and 22222 would have equalled $2\,(4^4) + 2\,(4^3) + 2\,(4^2) + 2\,(4) + 2$.

And because that which we now call 37, or $(3 \times 10) + 7 = (2 \times 16) + 4 + 1$, or $2\,(4^2) + 4 + 1$;

Therefore 37, on the decimal scale, might have been expressed as 211 on a quaternary scale, or as $(2 \times 4^2 + 4 + 1)$.

And whereas in considering ordinary numbers we break them up into tens, hundreds, and thousands, we must, if 4 were the base of our notation, decompose every number into fours and powers of four.

The following table will show how the written characters to represent the first twenty numbers might have been varied if the basis of a system of notation had been either two, six, eight, or twelve.

Decimal Scale.	Binary Scale (base 2).	Senary Scale (base 6).	Octary Scale (base 8).	Duodecimal Scale (base 12).
1	1	1	1	1
2	10	2	2	2
3	11	3	3	3
4	100	4	4	4
5	101	5	5	5
6	110	10	6	6
7	111	11	7	7
8	1000	12	10	8
9	1001	13	11	9
10	1010	14	12	a
11	1011	15	13	b
12	1100	20	14	10
13	1101	21	15	11
14	1110	22	16	12
15	1111	23	17	13
16	10000	24	20	14
17	10001	25	21	15
18	10010	30	22	16
19	10011	31	23	17
20	10100	32	24	18

Suppose it is required to find how 500 would have been written had either 4 or 7 been the base of our notation; it only becomes necessary to divide it into fours and sevens in the following manner:—

4)500
4)125·0
4)31·1
4) 7·3
1·3
13310

7)500
7)71·3
7)10·1
1·3
1313

13310 on the quaternary scale = 500 on the decimal; for
13310
10 4
$300 = 3 \times 4^2 = 48$
$3000 = 3 \times 4^3 = 192$
$10000 = 4^4 = 256$
500

1313 on the septenary scale = 500 on the decimal; for
1313
3 3
10 7
$300 = 3 \times 7^2 = 147$
$1000 = 7^3 = 343$
500

Whatever number be chosen as the base, one less than that number of digits is always required. Hence if eleven, twelve, or any greater number than ten were chosen it would be necessary to have new characters. Suppose twelve were the base, 10 would mean twelve, *t* might mean ten, and *e* eleven; then 25*e*7*t* would equal $(2 \times 12^4) + (5 \times 12^3) + (11 \times 12^2) + (7 \times 12) + 10$.

The following examples will serve to show how any one of the four fundamental operations of arithmetic might be performed on any scale which might be chosen.

I. Find 13 + 213 + 402 + 12 on the quinary (5) scale.

Scale of five.		Scale of ten.
13	=	8
213	=	58
402	=	102
12	=	7
1200		175

II. Subtract 142*t* from 27*e*9 on the duodecimal scale.

Duodecimal.		Decimal.
27*e*9	=	4605
142*t*	=	2338
138*e*	=	2267

III. Multiply 12212 by 7 on the ternary scale.

Ternary.		Decimal.
12212	=	158
21		7
12212		
102201		
1111222	=	1106

IV. Divide 2648 by 7 on the nonary scale.

Nonary.		Decimal.
7)2648	=	7)1988
345	=	284

EXERCISE CL.

1. Convert 52364 into the scale whose base is 5.

2. How would 10000 be expressed if the radix were 2, 7, or 11?

3. What number in the decimal scale equals 123421 on the quinary?

4. If 23454 be a number on the septenary scale, how will it be expressed on the scale whose radix is 8?

5. Add together 123, 432, 310, and 212 on the scale whose base is 6; multiply the result by 14, and give the answer in the duodecimal scale.

6. In what scale of notation will the number 37 be expressed as 122?

7. Convert 7423 in the nonary scale to an equivalent expression in the duodecimal.

8. Express 24738 respectively in the scales of 8, 9, and 11.

9. Make a table, showing how the first hundred ordinary numbers would be expressed if the scales were 3, 7, or 11.

The figures 1, 2, 3, &c., which are in common use in England, and in most other parts of Europe, are usually called *Arabic.** Their use became familiar among Arabic writers on Mathematics and Astronomy in the tenth century. But they were first employed in Europe by the Arabs, or Moors, who during several centuries occupied Spain, and did much to diffuse throughout Western Europe a love of calculation and of science.*

It is generally believed, however, that these symbols were used long before by the Hindoos, and that the Arabs learnt them of that people. They were brought into general use in Italy by Pope Sylvester II., but were not universally adopted throughout Europe till the fifteenth century. In several other nations the letters of the alphabet have been used to represent numbers; thus the Jews employed the first ten letters of the Hebrew alphabet to stand for the first ten numbers, and the remaining letters to stand for different collections of tens.

A similar plan of notation was adopted by the Greeks. Their alphabet contained 24 letters; to these three other characters were added, and thus, for the purposes of notation, they possessed 27 characters, or 3 nines. Of these the first nine were used for the nine units (from 1 to 9); the second to represent the nine tens (10 to 90); and the third for the nine hundreds (from 100 to 900). By combining these characters it was very easy to express any number up to 999. Higher numbers were represented by placing a point or dash under any one of them, and thus it was multiplied by 1000. Hence $\beta = 2$, but $\underset{,}{\beta} = 2000$. The letter μ (the initial of *myrias*, or 10,000) placed under any letter was used to multiply it by 10,000, thus $\beta_{\mu} = 20{,}000$. This system possessed many advantages, its chief defect being the absence of a cipher.

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* "The Arabic characters are found in Spanish manuscripts of the 12th century, in an Italian book written in 1220, and in a treatise, by John de Sacro Bosco, published about 1232 and entitled, 'De Algorismo.' Algorismus was the proper name for the Arabic notation and method of reckoning.

'In 1282 we find a single Arabic figure 3 inserted in a public record, the first indisputable instance of the employment of Arabic numerals in England."—*Hallam.*

The earlier notation in use among the Greeks was much more cumbersome. Simple strokes were used for the first four numbers, as, I, II, III, and IIII, but for five they used the first letter (Π) of the word *pente*, five; for ten the first letter (Δ) of the word *deka*, ten; for ten times 10 or one hundred, the first letter (H) of the word ἑκατὸν, written *Hekaton;* and for a thousand the first letter (X) of the word *chilia* = a thousand. Five times any number was expressed by putting a mark like the letter π over that number, thus $\overline{\vert\,\text{III}\,\vert}$ meant five times 3: $\overline{\vert\,\Delta\,\vert} = 50$, or 5 times 10; $\overline{\vert\,\chi\,\vert} = 5 \times 1000$, or 5000.*

The only one of the ancient systems of notation which we ever employ is that which was used by the Romans, and which is still painted on clock-faces, and placed at the heads of chapters in English books. The following table will show the several varieties of form which this singular system included.

1.	I.	16.	XVI.	400.	CCCC.
2.	II.	17.	XVII.	500.	D, IↃ.
3.	III.	18.	XVIII, XIIX.	600.	DC, IↃ C.
4	IIII, IV.	19.	XVIIII, XIX.	700.	DCC, IↃ CC.
5.	V.	20.	XX.	800.	DCCC, IↃ CCC.
6.	VI.	30.	XXX.	900.	DCCCC, IↃCCCC.
7.	VII.	40.	XXXX, XL.	1000.	CIↃ, M, $\bar{\text{I}}$.
8.	VIII, IIX.	50.	L.	2000.	CIↃ CIↃ, IICIↃ, IIM.
9.	VIIII, IX.	60.	LX.	5000.	IↃↃ, $\bar{\text{V}}$.
10.	X.	70.	LXX.	10000.	CCIↃↃ, $\bar{\text{X}}$.
11.	XI.	80.	LXXX, XXC.	50000.	IↃↃↃ, L.
12.	XII.	90.	LXXXX, XC.	100000.	CCCIↃↃↃ.
13.	XIII, XIIV.	100.	C.	500000.	IↃↃↃↃ.
14.	XIIII, XIV.	200.	CC.	1000000.	CCCCIↃↃↃↃ.
15.	XV.	300.	CCC.		

The principle of determining the value of a figure by its local position is only applied in a very limited way here. V stands for five, but IV means one less than five, while VI means one more than five, or six. In like manner, L expresses fifty, and C a hundred, but XL means ten less than fifty; LX ten more than fifty; XC a hundred all but ten; CX a hundred and ten.* That is, *whenever a character representing any number stands to the left of one representing a larger number, the value of the first is meant to be taken from that of the second; but whenever the characters are otherwise arranged, the separate values of each are to be added together.*

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* We have two familiar examples of the use of an entire alphabet for numbering. The 119th Psalm consists of 22 portions, which are distinguished by the 22 letters of the Hebrew alphabet. Each of Homer's great poems, the Iliad and Odyssey, consists of 24 books, and these are also distinguished by the 24 Greek letters. But this method of notation was of late date, and was seldom used.

Much difficulty has been felt in accounting for the choice of the letters V, X, D, M, &c., for the Roman notation. It has been conjectured that the first nine numbers were represented by simple strokes, thus, I, II, III, IIII, &c., and the ten at first by nine strokes with a bar across them ( ), and afterwards, for convenience, by the simple cross (×). Two strokes being thus required for ten, three were used for a hundred, thus, ⊏ or C, and four for a thousand, thus, M, or as found in old MSS. ↀ. Now the half of X is V,† the half of ⊏ is ∟. and the half of ↀ was originally written IↃ, and is easily contracted into D. It is very probably, however, that the C and M were chosen because they are the initials of the words *centum*, a hundred, and *mille*, a thousand.

In point of practical utility and convenience this system is far inferior to that of the Greeks. It is only fitted to register large numbers, and computation is almost impossible with it. But the Romans paid very little attention to Arithmetic as a science, or indeed to any branch of Mathematics. They used numbers for business transactions only; and their calculations, owing to their defective notation, were very laborious, mechanical, and involved. Slaves were kept in most large establishments, for the express purpose of keeping accounts, and they appear to have used either a *loculus*, a box of pebbles, or an *abacus*, a sort of frame with a number of wooden balls which could be arranged in columns to register units, tens, and hundreds.

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* In some cases this method of subtraction is carried further, thus IIX has been found used for 8, or 10—2, XXC for 80: and in some old German manuscripts IC is used for 99. This plan of reckoning by defect instead of addition is quite peculiar to the Roman system.

† V was the old form of writing the vowel *u*, which is the *fifth* of the series of vowels. Some writers attribute the use of V or five to this circumstance, but there is no proof that the coincidence is more than accidental.

MISCELLANEOUS QUESTIONS.

1. How many pieces of cloth 9 yards 2 qrs. 3 nails long can be cut out of a piece 52 yards 1 qr. 1 nail in length?

2. Find the value of 227 qrs. 3 bush. 2 pecks of wheat, at 36s. 8d. per quarter?

3. How many ounces of silver, at 5s. 6d. per ounce, are equivalent in value to 6 oz. 12 dwt. of gold, at £3 17s. 10½d. per oz.?

4. The sun's diameter is 111·454 times the equatorial diameter of the earth, which is 7925·648 miles. Required the sun's diameter in miles?

5. A manufacturer having a capital of £5,000, on which he can realize by hand labour 10 per cent. profit, buys a machine for £1,000, by which the profit on the remainder of his capital is raised to 20 per cent. This machine lasts 5 years; how much is he by that time the gainer, supposing him to draw £300 a year for the support of his family?

6. Find the value of—

   (*a*) 4 cwt. 2 qrs. 16 lbs. at £3 17s. 8¾d. per cwt.
   (*b*) 3 oz. 17 dwt. 20 grs. at 8s. 6¼d. per oz.

7. In the Julian correction of the calendar, every fourth year (leap year) consists of 366 days; in the Gregorian correction leap year is omitted three times in four centuries, but otherwise retained. Compare the mean lengths of the year according to those corrections respectively with the true length, which is 365 days 5 hours 48 min. 49·7 sec. nearly.

8. By the use of the Julian correction, how many superfluous days would have been introduced from the year 1 to the year 1750 A.D.?

9. Having bought goods for £20, I sell half of them so as to gain 10 per cent.; for how much must I sell the remainder so as to gain 20 per cent. on the whole?

10. Having bought goods for £18, I sell them four months afterwards for £25; what is the gain per cent. per annum?

11. Find the area of a floor whose length is 13 ft. 7 in. and its breadth 11 ft. 9 in., and show by means of a diagram what surface is represented in each term of the answer.

12. Between 1801 and 1811 the population of Edinburgh increased by 24¾ per cent., and in the latter year it was 102,987. What was it in 1801?

13. Show that any whole number, 765,468, may be expressed by the sum of its digits multiplied by powers of 10. Prove also by general reasoning that a number is divisible by 9 when the sum of its digits is divisible by 9, and only then.

14. Add together the fractions $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{19}{36}$, and explain why they must be first reduced to a common denominator.

15. What fraction of the earth's diameter (7,900 miles) is a mountain $4\frac{1}{2}$ miles high? By what fraction of an inch would the height of such a mountain be properly represented on a globe of 18 inches diameter?

16. Find the interest on £7,650 10s. for 5 years, at $3\frac{1}{2}$ per cent.

17. Multiply 2·564 by ·047, and divide ·00169 by ·013; verify your results by putting the decimals in the form of vulgar fractions.

18. Reduce $\frac{17}{625}$ to a decimal, and explain why, in reducing a fraction to a decimal which terminates, the number of decimal places depends on the form of the denominator of the fraction, and not on that of the numerator.

19. Extract the square root of 258368·89.

20. When are magnitudes (1) in arithmetical, (2) in geometrical progression? Find an arithmetical and a geometrical mean between a and b, write down the 12th term of the series 7, 12, 17, and find the sum of five terms of the series $\frac{1}{3} + \frac{2}{9} + \frac{4}{27}$, &c.

21. If 9,000 persons, travelling each 20 miles weekly, give a railway company a receipt of £900 in one week, how many persons travelling each 30 miles weekly will give a receipt of £62,400 a year, when the cost of travelling per mile is reduced one-half?

22. How many yards of carpet, 2 ft. 11 in. wide, will it take to cover a square floor, one side of which is 19 ft. 7 in.?

23. A tradesman having bought 200 eggs at 2 for a penny and 200 at 3 for a penny, sold the whole at 5 for twopence, how much did he gain or lose by the transaction?

24. Find the interest on £286 from the 1st of June to the 15th of September, at $3\frac{1}{2}$ per cent. per annum?

25. A, B, and C are joint owners of a ship; C's share is worth £400, A's share is $\frac{1}{3}$ of B's, and the sum of their shares is six-eighths of the value of the ship. Find the value of the shares held by A and B.

26. A person who has two-fifths of a ship sells three-fourths of his share for £1,500; what is the value of the share which remains to him, and what is the value of the whole ship?

27. Reduce 1 cwt. 3 qrs. 5 lbs. to the decimal of $\frac{3}{8}$ of a ton.

28. I borrow £130 on 5th March, and pay back £132 10s. 6d. on 18th October; what rate per cent. per annum of interest have I paid?

29. I wish to measure a distance of three furlongs with a line three rods and a half in length; how many times will the line measure the distance?

30. Add together the circulating decimals 0·53434, &c., and 0·465858, &c., and subtract the sum from $1\frac{1}{3}$.

31. Perform the operations indicated below :—

(*a*) 36·01 − 2·987564 (*d*) 6·25 ÷ 000125
(*b*) 2·745 × 45·674 (*e*) $\sqrt{2119 \cdot 6816}$
(*c*) 233·8268 ÷ 3·46.

32. Find the value of ·33333 of 2½ guineas.

33. Sum the series 4 + 11 + 18 + · · · · to 9 terms,
And also the series 3 + 6 + 12 + · · · · to 16 terms.

34. If 15 men, 12 women, and 9 boys can complete a piece of work in 50 days, how long would 9 men, 15 women, and 18 boys be in doing double the work; the parts done by each man, woman, and boy respectively, in the same time, being as the numbers 3, 2, 1?

35. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off, making way at the rate of 8 miles an hour; how many miles can the ship run before she will be overtaken?

36. If 12 men build a wall 60 ft. long, 4 ft. thick, and 20 ft. high, in 24 days, working 12 hours a day, how many must be employed to build a wall 100 ft. long, 3 feet thick, and 12 feet high, in 18 days, working 8 hours a day?

37. Find the sum of $\frac{3}{7}$ of a guinea, $\frac{5}{8}$ of a pound, and $\frac{7}{20}$ of a crown.

38. Place the first term in the following proportion:—

$$x : (15 + 9 - 2)^2 :: 5 \times 3 \times 2^4 : 6^2 + (6 \times 2^2).$$

39. If a spoon weigh 15 dwt. 11 grs., how many dozen of such spoons can be formed out of 122 oz. 9 dwt. 1 gr.?

40. A man spends at the rate of 12 guineas in 35 days, and saves £100 a year; what should be his annual income?

41. Required the number of square feet there are in a piece of slate 2¼ ft. and ⅓ in. long, and 1⅓ ft. and ½ in. in width.

42. What is the cost of the flooring of a schoolroom consisting of 36 planks, each plank 10½ ft. long, 8 in. wide, and 3 in. thick, a cubic foot being worth 1s. 7½d.?

43. There are 10 windows in a house; each window contains 12 panes of glass; what is the value of the whole glass, each pane being 1⅓ ft. long, 10 in. wide, and ⅕ in. thick, assuming that a cubic inch of glass is worth 2¼d.?

44. A boy loses ⅓ of his marbles; he plays again, and wins $\frac{4}{7}$ of ¾ of what he has left; he now has 80 marbles: how many did he have at first?

45. Divide £15 4s. 6d. among 4 men, 5 women, and 6 children; give the men each a share, the women each ¾ of a share, and the children each ⅔ of a share.

46. Solve the following expressions:—

$$\frac{\dfrac{2\frac{1}{3} \text{ of } 1\frac{1}{4}}{3\frac{2}{5} \div 7\frac{5}{6}}}{\dfrac{4\frac{4}{5} \text{ of } 6\frac{1}{27}}{1\frac{1}{18} + 4\frac{4}{9}}} \times \frac{3\frac{4}{7}}{3\frac{8}{14}} \; ; \qquad \frac{\dfrac{6 - \frac{2}{7}}{11 + \frac{3}{7}} \times \dfrac{6\frac{1}{8} \div \frac{147}{512}}{19\frac{1}{5} \times \frac{1}{14}}}{\dfrac{\frac{18}{20} \times \frac{3}{7} \text{ of } \frac{14}{19}}{\frac{5}{16} \text{ of } \frac{4}{11} \div \frac{8}{19}} \div \dfrac{\frac{7}{12} \times \frac{3}{5}}{\frac{11}{8} + \frac{4}{13}}}$$

47. If the sun shining on the ocean 7 hours per day causes 8756·75 cubic feet of water to evaporate per hour on a surface of 2·875 square miles, what would be the weight of water evaporated on a surface of 879·398 square miles in 6 days $2\frac{1}{2}$ hours, supposing the sun to shine 9 hours 37 minutes per day, and the weight of a cubic foot of water to be $62\frac{1}{2}$ lbs.?

48. A stage-coach has the circumference of its fore wheels $4\frac{3}{8}$ ft., and the circumference of the hind ones $12\frac{4}{19}$ ft. It is required—1. To find the number of revolutions performed by the fore and hind wheels in travelling $\frac{1}{7}$ of $\frac{1}{16}$ of the circumference of the earth (taken equal to 24,900 miles). 2. If the hind wheels make 5 revolutions every two seconds, how many revolutions do the fore ones perform per second? 3. In what time will the distance be travelled over according to this rate?

49. A colonist procures 9,000 acres of land from Government—$\frac{1}{3}$ of which is to be grazing land, $\frac{3}{7}$ arable land, and the remainder for the produce of hay—upon the following conditions; viz., that he is to give to the Government $\frac{2}{3}$ the annual profit of the grazing land, $\frac{1}{4}$ the annual profit of the arable land, and $\frac{1}{2}$ of the third kind of land. Now the whole annual profit from the first is £1 per acre, from the second £$1\frac{3}{8}$ per acre, and from the third £$\frac{9}{13}$ per acre; required the whole annual profit of the farm, and the sum due to each of the parties.

50. A ship at sea is known to sail at the rate of 10 miles per hour when the tide is with her; on the tide returning, her rate of sailing is reduced $\frac{1}{3}$ the former rate; after sailing for some time at this rate, the wind increasing, her speed is increased $\frac{1}{5}$ the last rate: required the distance travelled over in 12 hours, supposing her to sail $\frac{1}{3}$ of the time as in the first case, $\frac{2}{3}$ of the time as in the second case, and the remainder as in the third case.

51. If I transfer £7,280 from one kind of stock which is at 69·25, and pays 3 per cent., to another kind of stock at 108·6, which pays 5 per cent., what will be the difference in the annual income arising from the investment?

52. If I have to pay a bill of £370 at 3 months' date from this time, and I pay £120 of this sum at once, what extension of time ought to be allowed for the payment of the remainder?

53. Find the least number which is divisible by 9 without a remainder, and which, when divided by 7, leaves a remainder 4.

54. Having obtained the first five digits of the decimal equivalent to $\frac{1}{17}$ by actual division, deduce from the result the complete recurring period for $\frac{6}{17}$.

55. How much stock can be purchased by the transfer of £2,000 stock from the 3 per cents., at 90, to the $3\frac{1}{2}$ per cents., at 96; and what change will be effected in income by it?

56. A ship having a crew of 26 persons, carries provisions for 21 days; after having been at sea for 11 days they pick up a party from a wreck, and it is then found that the provisions will be exhausted in the course of 5 days: find the number of persons taken from the wreck.

57. I paid £28 per cwt. for tea, and sold it again for £32 10s. per cwt.; what quantity must be sold to gain £169 17s. 6d.?

58. A lump of iron, containing 11 cubic feet, is drawn out into a rod 14 yards long; what will be the thickness of the rod?

59. A contract is to be finished in 5 months 17 days, and 43 men are put on to work at once: at the end of ⅖ of this time, it is found that only ⅐ of the work is done; what extra number of hands will be required to complete the contract in the given time, the last employed men to work 12 hours per day, whilst the first 43 men work until the contract is completed only 10 hours per day?

60. "The quotation of gold at Paris is about 5 per mille premium, according to the new tariff, which, at the English Mint price of £3 17s. 10½d. per ounce for standard gold, gives an exchange of 25·29; and the exchange at Paris on London, being 25·45, it follows that gold is about 0·61 per cent. dearer in London than at Paris."

Explain and verify the above statement; it being given that 12 ounces of standard gold are equal in value to 11 ounces of fine gold, and that the new tariff price of a kilogramme of fine gold in Paris is 3437·75 francs, a kilogramme containing 32·154 English ounces.

61. Extract the square root of 82·1489.

62. If I buy goods worth £780, and sell them six months hence for £1,100, what will be my gain per cent. per annum, taking into account that the interest of money is 5 per cent.?

63. Resolve 54180 into its prime factors.

64. Show that any number will be divisible by 12 if its last two digits be divisible by 4, and the sum of its digits be divisible by 3 also.

65. A person placed £350 in a bank; to what sum will this amount in 6 years 10 months at 4½ per cent.?

66. If ⅓ + ¼ + $\frac{1}{12}$ of a number amount to 36, what is the number?

67. At what price must linen, which cost 2s. 10½d. per yard, be sold to gain 5½ per cent.?

68. Find the sum of £·775, ·824s., and ·305 of a crown, and express it in decimals of a guinea.

69. Explain the terms *interest* and *amount*, and find the interest of £111 11s. 11¾d. for 197 days, at 3¼ per cent.

70. I buy goods for £600, ready money, and sell them directly for £680, giving 3 months' credit: what is gained per cent. per annum?

71. What weight rests on the foundation of a pier, it being composed of 240 large and 300 small blocks of Yorkshire stone: each large block is 4 ft. 6 in. long, 2 ft. 6 in. broad, and 2 ft. thick; each small block is 3 ft. 6 in. long, 2 ft. 3 in. broad, and 2 ft. thick; and the weight of one cubic foot of the stone is 152 lbs.?

72. Extract the square root of 7 to five places of decimals, and the cube root of 517 to four places.

73. A shopkeeper who sells sugar which costs him £2,000, in a year, at a profit of 10 per cent., and tea which costs him £1,000, at a profit of 20 per cent., finds at the beginning of the next year that he must reduce the profit on his tea 5 per cent. By how much per cent. must he raise the price of his sugar to cover the loss, supposing him to sell tea and sugar of the same cost in that year?

74. There is a division of the labour of a certain manufacture between two sets of men, neither of which can do the other's work. The one set consists of 3 men and the other of 5, and when they work in this proportion both sets are just fully employed. One man of the first set stays away for a week; by what fraction are the earnings of each man thus diminished, supposing them to work by the piece, and to divide their earnings equally?

75. What will be the rent of a house for 3 months 2 weeks and 14 days, at £4 14s. 6d. per month?

76. If tea be purchased at 2s. 9d. per lb., and sold at 3s. 4d., what is the gain per cent.?

77. At what rate will the interest on £102 10s. amount to £12 13s. $8\frac{1}{4}$d. in $2\frac{3}{4}$ years?

78. Multiply $15\frac{5}{8}$ by $109\frac{5}{7}$, and divide £61 4s. $7\frac{5}{16}$d. by $267\frac{3}{16}$.

79. If £15 12s. pay 16 labourers for 18 days, how many labourers at the same rate will £62 8s. pay for 24 days?

80. Reduce 2s. $7\frac{1}{2}$d. to the decimal of £3, and 2 hours 16 minutes to the decimal of a day.

81. Find the interest on £436 10s. for 3 years 8 months at £2 13s. 6d. per cent.

82. How many cubic feet of water does a ship displace whose gross weight is 1,500 tons, each cubic foot of water weighing $62\frac{1}{2}$ lbs.?

83. Subtract $\frac{3}{7}$ from 1·1, and divide the remainder by 0·1.

84. Why must the decimal equivalent to $\frac{5}{7}$ recur? Find that decimal. Find the vulgar fractions equivalent to the recurring decimals—

$$\cdot 71717171\dot{7}\dot{1} \quad \text{and} \quad \cdot 80654\dot{6}5\dot{4}.$$

85. Show that in a system of logarithms to the base of 10, the logarithms of all numbers between 1 and 10 must be included between 0 and 1. How is the characteristic of a logarithm generally determined? Given log. 2 = ·3010300, log. 3 = ·4771213, find log. 80, log. 81, log. 360.

86. The area of England being 50535 square miles, find the number of acres which it contains.

87. What is the value of 5487 cwt. at £1 17s. $3\frac{3}{4}$d. per cwt.?

88. If 27 men cut down 108 acres of grass in 5 days, working 12 hours a day, how many acres would 16 men cut down in 9 days, working 12 hours a day?

89. Suppose an ear of wheat to contain 187 grains of corn, what would be the produce of half an acre if there be 13 ears to every square inch?

90. Suppose I send 1857 quarters of wheat to Australia, and when sold there at 73s. 6d. per quarter, it realizes a profit of £$15\frac{3}{7}$ per cent., what sum should I receive for it, and what would be my profit? and suppose also that I receive gold dust in exchange at the rate of £3 per ounce there, and from fluctuation in the money market it is worth here £3 12s. 9d., what will be my gain in the transaction?

91. Divide a guinea between A, B, C, and D, so that B's share is ⅛ more than A's, C's ⅛ more than B's, and D's ⅛ more than C's.

92. Suppose 5·434 lbs. to cost seven-tenths of £3 11s. 6d., what will 17 tons 14 cwt. 3 qrs. 15 lbs. 6 oz. cost at the same rate?

93. A person travelled altogether 805 miles, of which he went 8 by water to 5 by railway, and 7 by water to 3 on foot; how far did he go in each way?

94. I commence business with the sum of £500, and at the end of 15 years find I have realized the sum of £14,756; at what rate per annum have I gained that sum, supposing I receive ⅛ more profit each succeeding year after the first 4 years?

95. A person exports goods to the amount of £3,754 15s. 9d., for which he agrees to take ivory at 1s. 7¾d. per lb., which he sells again, realizing a profit of £$17\frac{3}{17}$ per cent.; what profit does he make?

96. The side of a cube is 13 ft. in length; what is the side of a cube three times its size?

97. How many imperial gallons, each 277·274 cubic inches, would be contained in a trough, 18 ft. 6 in. long, 2 ft. 4 in. wide, and 1 ft. 5 in. deep?

98. What would be the cost of a steam engine, supposing that the materials consist of 19 tons 15 cwt. 3 qrs. 11 lbs. of iron, at £21 16s. 11¾d. per ton; brass, 5 cwt. 3 qrs. 18 lbs. 4 oz., at 1s. 10½d. per lb.; other materials representing £47 11s. 4d. The men's time employed at the manufacture being—engineers, 1,475 hours at 5s. 10d. per day; smiths' time, 436 hours, at 6¾d. per hour; coppersmiths' time, 176 hours, at 6¼d. per hour; and ironfounders, 275 hours, at 5s. 11d. per day?

SPECIMENS OF ARITHMETICAL PAPERS SET AT THE OXFORD AND CAMBRIDGE LOCAL EXAMINATIONS.

OXFORD.

SENIOR STUDENTS I.

[N.B.—Every candidate is required to satisfy the examiners in this paper. Attention should be paid to spelling and handwriting.

No credit will be given for any answer, the full working of which is not shown.]

1. How many pounds are there in 91,200 farthings?

2. Divide £3,050 9s. 10½d. by 81.

3. Find, by Practice, the value of 245 things at £3 19s. 9¾d. each, and of 7 cwt. 3 qrs. 26 lbs. at £1 10s. 4d. per cwt.

4. Simplify $\frac{261}{3103}$; and $\dfrac{3}{8-\dfrac{7}{2-\frac{3}{4}}}+\dfrac{5}{6-\dfrac{5}{2-\frac{5}{6}}}$.

5. Add together $\frac{2}{3}$ of a crown, $\frac{13}{126}$ of a guinea, $\cdot\dot{3}$ of 18s. 6d., and $\cdot 41\dot{6}$ of £1.

6. Divide ·024 by 60, 24 by ·006, and 2·4 by ·06.

7. Express as vulgar fractions, ·375, ·0375, and $\cdot 0\dot{1}0\dot{9}$.

8. What vulgar fraction and what decimal fraction is 1½ ft. of ¼ of a mile?

9. Extract the square root of 49196196, and of ·0016.

10. Compare the simple and compound interest on £119 at the end of three years at 4 per cent. per annum.

11. How many sovereigns are there in 80 lbs. of standard gold, an ounce of standard gold being worth £3 17s. 10½d.?

12. How many planks, each 13½ feet long and 10½ inches wide, will be required for the construction of a platform 54 yards long and 21 yards broad? What will be the cost at 5½d. per square foot?

13. If 5 horses eat 8 bushels 1¾ pecks of oats in 9 days, how long, at the same rate, will 66 bushels 3¾ pecks last 17 horses?

SENIOR STUDENTS II.

1. Find the number of pounds (Troy weight) in two millions three hundred and fifty thousand and eighty grains, and divide ten shillings among sixty persons.

2. Divide one thousand five hundred and forty-nine pounds nineteen shillings and fourpence three farthings by thirty-one, and multiply one hundredweight twenty-seven pounds fifteen ounces by seventeen.

3. Find, by Practice, the value of three hundred and nine things at two pounds eighteen shillings and fourpence each; and the cost of seven hundredweight two quarters twenty-four pounds at one pound seven shillings and fivepence a hundredweight.

4. Simplify $\frac{377}{403}$ and $\frac{\frac{2}{9} \text{ of } 1\frac{2}{3} \text{ of } 2\frac{2}{5}}{\frac{7}{9}+\frac{5}{12}-\frac{3}{4}}$.

5. Add together $\frac{1}{5}$ of half a crown, $\frac{1}{6}$ of a guinea, $\frac{4}{15}$ of £1, and $\cdot 01\dot{3}$ of £1 5s.

6. Divide ·144 by 1·2, 144 by ·6, and ·144 by ·016.

7. What part of a pound is eightpence? and what decimal fraction s a scruple of a pound?

8. Cube 1·04, and extract the square root of 870014016.

9. Compare the simple and compound interest on £10 8s. at the end of three years, reckoning money at 4 per cent. per annum.

10. Which is the better investment, the 4 per Cents. at 120, or the $2\frac{1}{2}$ per Cents. at 75?

11. If £4 14s. $10\frac{1}{2}$d. buy 288 yards, how many can be bought for £1 11s. $7\frac{1}{2}$d.?

12. If 70 men dig 1 acre 2 roods 10 perches in 5 days, how many men will dig $2\frac{1}{2}$ acres in 28 days?

SENIOR STUDENTS III.

1. Write out Troy weight. How many grains are there in a pound avoirdupois? How many pounds avoirdupois are equivalent to one hundred and nine thousand three hundred and seventy-five pounds Troy?

2. Divide two hundred and eight pounds two shillings and sixpence farthing by twenty-three, and multiply one mile seven furlongs thirty-nine poles by forty-one.

3. Obtain, by Practice, the cost of three hundred and fifty-five things at one pound sixteen shillings and eightpence each, and calculate a person's wages for five months three weeks and six days at one pound seven shillings and fivepence per month.

4. Simplify $\frac{215}{440}$, and $\frac{5}{12}$ of $\frac{9}{16}$ of $2\frac{2}{11}$. Also add the results.

5. Add together $\frac{3}{8}$ of £1 6s. $6\frac{3}{4}$d., $\frac{1}{336}$ of a guinea, $\frac{5}{64}$ of a sovereign, ·4375 of a shilling, and ·13 of half a sovereign.

6. Divide ·045 by ·0015, 4·5 by 150, and ·45 by ·15.

7. What part of a sovereign is threepence three farthings? and what decimal fraction of a pole is an inch?

8. Extract the square root of 893830609, and raise 1·05 to the fourth power.

9. Compare the simple and compound interest on £21 10s. at the end of four years, reckoning money at 5 per cent. per annum.

10. If a farthing be the interest on a shilling for a calendar month, what is the rate per cent. per annum?

11. If twenty-seven hundredweight twenty-one pounds cost three hundred and seventy-nine pounds two shillings and threepence three farthings, what will be the cost of three hundredweight three quarters fifteen pounds?

12. If three persons are boarded four weeks for seven pounds, how many can be boarded thirteen weeks five days for one hundred and twelve pounds?

JUNIOR STUDENTS I.

1. Divide 64 by ·08, 6·4 by 80, and ·064 by ·008.

2. Express in vulgar fractions ·25, ·025, and $\cdot 1\dot{2}\dot{7}$.

3. Simplify $\dfrac{1}{1 + \dfrac{7}{6\frac{2}{3}}} + \dfrac{7}{13\frac{2}{3}}$.

4. Reduce $2\frac{1}{2}$ gills to the vulgar and decimal fractions of $3\frac{1}{3}$ gallons.

5. Extract the square root of 502681 and ·0009.

6. What is the value of 40 lbs. of standard gold if an ounce be worth £3 17s. $10\frac{1}{2}$d.?

7. Find the cost of papering a room 5 yds. 1 ft. $2\frac{1}{2}$ in. long, 5 yds. $3\frac{1}{2}$ in. broad, 4 yds. high, with paper 9 in. wide, at $2\frac{1}{2}$d. a yard.

8. A box is 4 ft. long, 2 ft. 6 in. wide, 1 ft. 6 in. deep. In it are packed 400 books, each $8\frac{3}{4}$ in. long, $5\frac{1}{2}$ in. wide, $1\frac{1}{3}$ in. thick. For how many more books, $6\frac{1}{3}$ in. long, 4 in. wide, and $1\frac{1}{4}$ in. thick, will there be room left?

9. A rectangular field is 400 ft. long and 300 ft. wide. What is its area in acres? Also find the areas of the portions into which it is divided by a line drawn from the middle point of one side to one of the opposite corners.

10. A square tower, 21 feet on each side, is to have either a flat roof covered with sheet lead which costs 6d. per square foot, or a pyramidal roof, whose vertical height is 10 feet, covered with slates, which cost 18s. 9d. per hundred, and each of which has an exposed surface of 12 in. by 9 in. Find the cost in each case.

11. The top of a circular table is 7 feet in diameter and 1 in. thick. How many cubic feet of wood does it contain, and what will it cost to polish its upper surface at 6d. a square foot?

12. How many cubic inches of iron are there in a spherical cannon ball 9 in. in diameter? If the ball is melted and cast into a conical mould the base of which is 18 in. in diameter, show that the height of the cone will be $4\frac{1}{2}$ in.

JUNIOR STUDENTS II.

1. Add together one million fifteen thousand and eighty, four hundred and nine thousand seven hundred and ninety, two hundred and forty-two thousand six hundred and ninety-nine, and subtract nine hundred and fifty-eight thousand seven hundred and ninety-one from the sum.

Explain the operation of "borrowing" in subtraction.

2. Multiply 17943 by 5079, and divide 348753392 by 688.

3. Add together £1047 16s. 4¾d., £1 0s. 9½d., 14s. 8¼d., £62 0s. 1¼d.; and subtract 5 tons 17 cwt. 2 qrs. 15 lbs. from 7 tons 14 cwt. 2 qrs. 10 lbs.

4. Reduce £147 17s. 6½d. to farthings, add ten thousand and forty-seven farthings to the result, and express the sum in £ s. d.

5. Write out the table of avoirdupois weight, and show that 144 lbs. avoirdupois = 175 lbs. Troy.

6. Multiply 18 tons 3 cwt. 2 qrs. 9 oz. by 23.

7. A sum of £99 2s. 9¾d. is equally divided among 47 people: what does each receive? And if each subscribe one-third of his share to a fund, what will be the total amount of their subscriptions?

8. The circumference of a carriage wheel is 7 ft. 4 in.: how often does it turn round in running a league?

9. If 15 yds. of silk cost £1 13s. 9d., how much will 20 yds. 1 ft. cost?

10. A man buys 500 quarters of wheat at 56s. a quarter; he sells one-half of this quantity at the rate of 6s. a bushel: at what rate must he sell the remainder so as to gain £25 by the whole transaction?

JUNIOR STUDENTS III.

1. Divide 1·69 by ·013, 16·9 by 13, and 169 by 1·3.

2. Add together $1\frac{1}{15}$ of half a crown, $\frac{7}{16}$ of £1, ·25 of £1 12s. 4d., and $\cdot 0\dot{3}$ of 15s.

3. Simplify $\dfrac{1\frac{3}{5}}{3+\dfrac{1}{3\frac{1}{3}}} + \dfrac{1\frac{2}{7} \text{ of } 4\frac{2}{7}}{1\frac{2}{9} \text{ of } 3\frac{6}{7}}$ and $\dfrac{5\frac{2}{5} \text{ of } 7\frac{2}{9}}{8\frac{7}{24} - 3\frac{5}{12}}$.

4. Cube 1·05, and extract the square root of 61013446081.

5. Compare the compound and simple interest on £10 10s. at the end of three years at 5 per cent. per annum.

6. If 20 per cent. be lost on a horse sold for £19 4s., what was the cost of the horse?

7. If a man earns 30s. in 8 days, working 9 hours a day, in how many days will he earn 50s., working 10 hours a day?

8. A room 39 ft. long requires 36 yds. of carpet 2 ft. 2 in. wide to cover it : what is the breadth of the room ?

9. A gravel path 4 ft. wide is made round a circular court 90 yds. in diameter. The making of the path costs 1d. per sq. yd., and a border along its inner edge costs 3d. a yd. Find the total cost.

10. A pyramid of lead is 14 in. high, and stands on a square base 6 in. long on each side. How many cubic inches of lead does it contain, and how many spherical bullets ·75 in. in diameter can be made out of it?

11. What will it cost to make a cylindrical barrel closed at both ends, 3 ft. high and 3 ft. in diameter, the price of wood being 6d. per sq. ft., and the charge for labour being made at the rate of 3d. a gallon on the contents of the barrel ? [Take a gallon = 270 cubic inches.]

12. A cylindrical stick $\frac{1}{3}$ in. thick is sharpened at one end into the shape of a cone, the slant side of which is $\frac{5}{18}$ in. long. How much wood is cut away in doing this ?

CAMBRIDGE.

SENIOR STUDENTS I.

1. Explain the meaning of the terms *unit*, *number;* and distinguish between *abstract* and *concrete* numbers.

2. £135 13s. $4\frac{1}{2}$d. is distributed equally amongst 42 poor people ; find the share of each.

3. Add together $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{1}{4}$, $4\frac{1}{5}$, $5\frac{1}{6}$; and subtract the sum from 17.
Simplify—

$$\frac{5}{21} \times \frac{13}{15} \times \frac{48}{109} \times \frac{105}{208} \times 21\frac{4}{5}; \quad \frac{9\frac{1}{4} - 2\frac{7}{10}}{2\frac{3}{4} + 3\frac{4}{5}}.$$

4. Find, *by practice*, the value of $343\frac{3}{7}$ things at £2 16s. $10\frac{1}{2}$d. each.

5. Subtract ·992748 from 1 ; divide the result by ·0074, and prove the truth of your division by vulgar fractions.

Multiply $9\cdot\dot{1}4285\dot{7}$ by ·109375.

6. Find the value of $\frac{3}{8}$ of £1 10s. + $\frac{2}{3}$ of $1\frac{1}{2}$ of $\frac{4}{5}$ of £1 5s. − ·5625 of £1 : and reduce 2 cwt. 3 qrs. 3 lbs. 8 oz. to the decimal of a ton.

7. Find the cost of papering a room of which the length is 28 feet 6 in., the breadth 18 feet 9 in., and the height 12 feet, .with paper 1 ft. 9 in. broad, at half a crown per yard.

8. A cistern is supplied with two feeding pipes, capable of filling it in half an hour and three quarters of an hour respectively : and one discharge pipe capable of emptying it in a quarter of an hour. The

cistern is full, and the three pipes are set into action together: what portion of the cistern will remain filled in three quarters of an hour?

9. Explain the meaning of *interest;* and distinguish between *simple* and *compound* interest. Show that the interest on £450 for 7 months at 4 per cent. per annum is equal to the discount on £460 10s. for the same time at the same rate per cent.: and explain why these two sums are equal.

10. An up train 88 yards long, travelling at the rate of 35 miles an hour, meets a down train 88 yards long, at 12 o'clock, and passes it in 6 seconds. At 15 minutes and 3 seconds past 12 the up train meets a second down train 132 yards long, and passes it also in 6 seconds. At what time will the second down train run into the first?

SENIOR STUDENTS II.

1. What is the meaning of 25, 2⅓ and 2·3?

In subtraction how do you get over the difficulty of taking a greater digit from a less? Illustrate your answer by taking 874 from 953.

2. Divide 15997 by 21 by short division. Give a reason for your method of determining the remainder.

3. Supposing unity to be represented by 3¾d., find the value of fifty millions five thousand and six.

How many tons, cwts., &c., are there in as many ounces as there are seconds in a week?

4. Add together—

$$\frac{7}{15},\ \frac{9}{17},\ \frac{13}{25},\ \frac{23}{24}.$$

Simplify— $$8\frac{1}{2} - 4\frac{2}{3} + \frac{3}{5} - 1\frac{1}{6} \text{ of } \frac{2}{5}:$$

and $$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}.$$

5. Find, by Practice, the value of 2037⅛ cwt. of soap at £1 19s. 4½d. a cwt.

6. Multiply ·0204 by 40·2: divide ·04 by 20, ·4 by ·002, ·0004 by ·0002, and 400 by ·02. Prove the truth of two of your results by means of vulgar fractions.

7. Find a decimal which shall be within $\frac{1}{10000}$ of $\frac{17}{42}$.

Find the value of $\cdot\dot{5}\dot{4}$ of $\cdot 30\dot{2}\dot{7}$ of 1 m. 6 fur. 12 po.

8. A closed vessel formed of metal 1 inch thick, whose external dimensions are 8 ft. 3 in., 7 ft. 5 in., and 4 ft. 3 in., weighs 3 cwt. 1 qr. 8 lbs. What would be the weight of a solid mass of metal of the same dimensions?

9. On a piece of work 3 men and 5 boys are employed, who do half of it in six days. After this one more man and one more boy are put on, and ⅓ more is done in 3 days; how many more men must be put on that the whole may be completed in one day more?

10. If 9 oz. of gold, 10 carats fine, and 5 oz., 11 carats fine, be mixed with 6 oz. of unknown fineness, and the fineness of the resulting mixture be 12 carats, what was the unknown fineness?

11. If 3 per cent. consols be at 90⅝, what sum must I invest in order to secure from them a yearly income of £470 after paying an income tax of 5d. in the pound, brokerage being at ⅛ per cent.?

12. Find the square root of 13 to five decimal places.

A cubical block contains 9 cub. ft. 1029 cub. in.; find the number of sq. yds., &c., in its surface.

JUNIOR STUDENTS I.

1. Subtract one hundred and seven thousand and ten from twenty millions ten thousand one hundred and one.

Divide the result by twenty-five.

2. A tax collector took at one house £1 0s. 1½d., at another £21 1s. 6d. at a third £6 8s. 0¼d., at a fourth £1 17s. 5½d.; on returning home his pocket burst and all the money he had collected was scattered on the ground; he picked up £30 7s. 0¾d.: did he lose any money? if so, how much?

3. If £102354 14s. 8¼d. be divided equally amongst 93 persons, how much will each receive?

4. Add together $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{1}{6}$. Find the value of:

$$\left(4\frac{2}{5} + \frac{7}{2}\right) \text{ of } \left(\frac{2}{7} + 1\right).$$

5. Find by practice the value of 1232 things at £1 3s. 4½d. each.

6. Divide 210720·6 by ·4206; and ·2107206 by 42·06.

What decimal fraction is £1 16s. 11d. of £33 4s. 6d.?

7. If 32 horses eat 96 bushels of corn in 21 days, for how many days will 66 bushels feed 7 horses?

8. If 3 ducks are worth 4 chickens, and 3 geese are worth 10 ducks, find the value of a goose; a pair of chickens being worth 4s. 6d.

9. What is the difference between Interest and Discount?

Find the simple interest and amount of £63 5s. 9d. for 10½ years at 3¼ per cent.

10. A tank is 20 ft. 9 in. long, 15 ft. 7 in. wide, and 6 ft. 4 in. deep; find how much water it will hold in cubic feet and inches.

11. A person invests £1365 in the 3 per cents. at 91; he sells out £1000 stock when they have risen to 93½, and the remainder when they have fallen to 85. How much does he gain or lose by the transaction?

12. On the 1st of October, 1870, a merchant's assets are £6500 in merchandise, £600 in cash, and £500 in debts due to him by *B*, and his liabilities are £350 due to *A*.

The following is the business of the month:—

Oct.		£
1	Paid cash for rent of premises	150
7	Goods sold to *B*	520 ✓
10	Cash to *A*	300
15	Goods bought of *A*	450
20	Cash from *B*	250 ✓
26	Wages	30
27	Goods bought of *A*	100
31	Goods sold to *B*	170 ✓

On October the 31st the merchandise on hand is valued at £6410. Post the ledger and determine the merchant's position at the end of the month.

JUNIOR STUDENTS II.

1. Write down two millions five hundred and two, and subtract from it nine hundred and nine thousand seven hundred and ten.

2. Reduce 527895 inches to poles.

3. Out of a sum of money, 17 men receive 3s. 4½d. each, and there remained 2s. 5¾d.; how much was the original sum?

4. How many times may £17 5s. 9½d. be subtracted from £100, and what will be the last remainder?

5. Add together $\frac{5}{17}$, $\frac{2}{51}$, and $\frac{3}{34}$, and divide the product of $7\frac{3}{4}$ and $9\frac{1}{31}$ by the difference of 9 and $6\frac{1}{5}$.

6. State your rule for division of decimals, and divide ·1 separately by ·5, ·0005, and 5000.

7. Find the value of 7339 lbs. at £5 3s. 9d. per lb.

8. If £100 will pay for the railway travelling of 20 travellers for 5 days of 12 hours each, how long would it last 16 travellers, travelling 6 hours a day, the rate of travelling being the same?

9. Taking 8000 metres to be equal to 5 miles, how many square metres are there in an acre?

10. Find the Simple and Compound Interest on £2000 for 3 years at 7½ per cent.

11. Three men, *A*, *B*, *C*, play a game, starting with equal sums; each stakes for each game ⅓ of all he then has. If *A* wins the first game, and *B* the second, what fraction of what they originally had has each left?

12. I buy 10 shares of £20 each at 27½; I receive a dividend of 15 per cent. and then sell out at 37½, how much have I gained in all?

13. John Smith begins a wine business. Jan. 1st he has £2000 cash, and wine to the value of £1000. The following are his transactions in January:—

Jan.		£	s.	d.
1	Advanced to Petty Cash	10	0	0
2	Bought wine of W. Day on credit	500	0	0
3	Sold wine on credit to H. Todd	75	0	0
5	Paid W. Day on account	200	0	0
10	Sold wine on credit to H. Todd	100	0	0
20	Received from H. Todd on account ...	50	0	0
...	Bought wine for ready money and paid cash	1150	0	0
31	Paid rent to *X*	20	0	0
...	Trade charges paid out of Petty Cash ...	7	3	4
...	Stock of wine on hand valued at	2535	0	0

Post the ledger, and make out a balance-sheet.

ANSWERS TO THE EXERCISES.

EXERCISE III.

Two thousand three hundred and five ; eight hundred and six ; seven thousand and ninety-five ; twenty thousand three hundred ; four hundred and fifty-seven thousand two hundred and ninety-eight ; six hundred and twenty-seven thousand four hundred and twenty-one ; thirty-three thousand nine hundred and eleven ; four hundred and twenty-seven thousand eight hundred and sixteen ; nine million and thirty-two thousand eight hundred and four ; eight million two hundred and seventy-one thousand and ninety-six ; thirty-two million seven hundred and forty-five thousand eight hundred and forty-one ; seventy-two thousand nine hundred and eighteen ; thirteen ; five thousand one hundred and seventy-two ; eight hundred and forty ; six hundred and twenty-one.

V.—(1) 980 ; 40002. (2) 7600 ; 80402. (3) 250 ; 5004007. (4) 8600 ; 24005. (5) 1215 ; 6410. (6) 981 ; 18000006. (7) 413 ; 500.

VII.—(1) 24917. (2) 17866. (3) 102119. (4) 9496. (5) 2977. (6) 7865. (7) 4961. (8) 17985. (9) 351152. (10) 27166. (11) 172846. (12) 60562. (13) 8309827. (14) 8970598. (15) 3028394. (16) 16548360. (17) 12461. (18) 54015674. (19) 21735. (20) Gold 283, silver 568, copper 954 : total 1805. (21) 1333, 1720. (22) Wheat 4368, barley 10298, oats 3042 : total 17708. (23) 3366, 2222, 5879, 4223, 2320 (in 1875). (24) 290 days. (25) £4597. (26) 4314 letters. (27) 10589009.

VIII.—(1) £2900 19s. 5½d. (2) £11100 15s. 7½d. (3) £3338 3s. 11d. (4) £1243 15s. 2½d. (5) 16 tons 16 cwt. 2 qrs. 5 lbs. (6) 41 lbs. 8 oz. 5 dwt. 17 grs. (7) 4 tons 3 cwt. 1 qr. 26 lbs. 7 oz. 4 drs. (8) 21 miles 6 fur. 33 poles 3 yds. 2 ft. 7 in. (9) 90 acres 2 roods 4 poles 30 yds. (10) 22 qrs. 5 bush. 3 pecks 1 gal. 3 qts. (11) 1 cwt. 3 qrs. 6 lbs. 1 oz. 6 drs. (12) 66 days 17 hours 10 min. (13) 300 acres 0 roods 38 poles. (14) 232 yds. 2 ft. 134 in. (15) 222 gals. (16) 30 hrs. 43 min. 42 sec. (17) 94 leagues 1 mile 7 fur. 12 poles 1 ft. (18) 382 yds. 1 qr. (19) 40 miles 7 fur. 6 poles 2 ft. 5 in. (20) £1076 4s. 5½d. (21) 1 qr. 15 lbs. 12 oz. 10 drs.

IX.—(1) 124; 263; 2112. (2) 44; 54; 252; 3444.

X.—(1) 6; 613; 707; 465; 8024; 49724. (2) 87208; 2563; 8297; 36747; 1789; 3611. (3) 241; 341; 5784; 22706; 3424; 5319. (4) 507; 157; 366; 1083. (5) 37393; 999194; 7055. (6) 37993; 2678; 22815; 1265; 47961; 11644; 37364; 50611; 39410. (7) 214; 52256; 25480; 12965; 36718; 1773; 497929; 5061. (8) 1959. (9) 71895; 16717. (10) £87332. (11) In 1875; the answers are, 1322; 1803; 818; 199; 1680. (12) Greater, 758865; difference, 549183. (13) First, 44; second, 59; third, 15; fourth, 12. (14) He is worth £317. (15) 19092. (16) 6775. (17) 31326; 12710; and 18616. (18) 143. (19) 92. (20) 149. (21) 22704108; and 623302.

XI.—(1) 4 years 581 days 40 hours 114 minutes; 26 weeks 9 days 28 hours. (2) 4 tons 22 cwt. 5 qrs. 38 lbs.; 46 lbs. 18 oz. 25 dwt. 29 grs. (3) 275 qrs. 10 bush. 5 pecks 3 gals.; 17 hours 61 min. 77 sec. (4) 10 yds. 4 ft. 20 in.; 1 mile 10 fur. 47 poles; 4 miles 1768 yards.

XII.—(1) £51 9s. 0¼d.; £108 17s. 3d. (2) £703 6s. 1d.; £51 9s. 7d. (3) £702 3s. 9d. (4) £4808 0s. 7½d.; £220 3s. 9d. (5) £226 15s. 6½d.; £3781 13s. 8¾d. (6) £2738 3s. 10d.; £2705 3s. 8½d. (7) £11927 18s. 11¼d.; £10 2s. 7d. (8) £62671 5s. 3¾d.; (9) 21 weeks 3 days 19 hours. (10) 16 miles 3 poles 3 yds. 1 ft. 6 in. (11) 4 leagues 1 mile 4 fur. 37 poles 4 yds. 1 ft. 6 in. (12) £1225 3s. 5¼d. (13) 12 cwt. 14 lbs.; 1 qr. 19 lbs. 3 oz. (14) 4 cwt. 1 qr. 16 lbs.; 5 lbs. 6 oz. 4 drs. (15) 5 miles 4 fur. 23 poles. (16) £513 12s. 11d. (17) 27 tons 13 cwt. 2 qrs. 12 lbs. (18) 79 acres 36 poles. (19) 3 leagues 1 mile 1 fur. 9 poles. (20) 18 yds. 2 qrs. 3 nails. (21) 6 qrs. 2 bush. 1 gal. 2 qts. (22) At 9·7 a.m. March 14, 1791. (23) £1499 1s. 10¾d. (24) 8 cwt. 22 lbs. 6 oz. 9 drs.; 16 tons 15 cwt. 3 qrs. 8 lbs. 9 drs. (25) 13 st. 7 lbs. 7 oz. 13 drs.; 13 st. 12 lbs. 5 oz. 9 drs. (26) 21 miles 3 fur. 19 poles 4 yds.; 7 yds. 1 ft. 9 in. (27) 19 acres 29 poles; 33 acres 1 rood 17 poles 27¾ sq. yds. (28) 2 roods 36 poles. (29) 30 years 29 weeks 2 days. (30) 20 min. 14 sec. (31) 8° 40′. (32) 6s. 9¾d. (33) 2 qrs. 21 lbs. 12 oz. 7 drs. (34) 4 miles 4 fur. 13 poles 5 yds.

XIII.—b (1) 9; 61. (2) 31; 3. (3) 182; 707. (4) 391; $x - 6$. (5) $a + 3$; $a - 5$.

XV.—(1) 18; 3; 79; 40. (2) $b-3$; $y+5$. (3) $x-7$; $n-x$. (4) 9; 9; $m+6$.

XVI.—(1) 1981; 253145; 11856; 26192. (2) 187623; 6798; 35209284. (3) 4374; 589659; 52776; 6411132; 117148. (4) 3684709; 1006530; 29110844; 5064402. (5) 3394992; 3474; 508374; 19820935. (6) 2482956; 6326883; 4369664; 2188396. (7) 226. (8) 1259. (9) 338. (10) 1728. (11) 19236. (12) 222. (13) 430.

XVII.—(1) 671216; 10152; 197280. (2) 3020256; 3059420; 229957; 53374167. (3) 23092488; 1702910; 373005; 85680. (4) 1036 miles. (5) 9630. (6) 2850. (7) 172320. (8) 540288.

XVIII.—(1) 262380; 594800; 24822000. (2) 39792900; 22246000; 26716050000. (3) 1749520000; 15736050000; 1433760000. (4) 23388400; 789920000.

XIX.—(1) 3906; 3525993. (2) 66710074; 7199076; 368244. (3) 6 395025; 313632; 141684375. (4) 29462508; 48954719. (5) 51098208; 4003868253; 6535022130. (6) 109157832; 209385353; 46194169884. (7) 22218444; 4768389; 19747892. (8) 49615636; 36630; 993720. (9) 11805528; 3601620. (10) 19686992; 355752; 111680275. (11) 22589658; 86208. (12) £256. (13) 7074. (14) 2549470. (15) 100368. (16) 2482. (17) 4770 feet. (18) 493314. (19) 170820.

XX.—(1) £35 18s. 4d.; £59 2s.; £17 5s. 2¼d. (2) £191 10s. 9d.; £313 19s. 4½d.; £1411 17s. 7½d. (3) £21967 11s. 8d.; £10435 13s. 1½d.; £656181 18s. (4) £534789 4s. 9½d.; £2901990; £3359986 5s. 8¼d. (5) £1602832 0s. 1½d.; £2793993 5s. 3d. (6) 97 acres 1 rood 15 poles; 38 tons 17 cwt. 1 qr. 12 lbs.; 63 miles 5 fur. 24 poles. (7) 485 miles 37 poles 4 yds.; 340 lbs. 8 dwt. 4 grs.; 257 lbs. 1 oz. 6 drs. 2 scruples. (8) 5 tons 3 cwt. 1 qr. 7 lbs.; 248 tons 15 cwt. 2 qrs. 14 lbs.; 101 tons 14 cwt. 1 qr. 2 lbs. (9) 1 lb. 3 oz. 9 dwt.; 11 lbs. 10 oz. 9 dwt. 12 grs.; 459 lbs. 4 oz. 5 dwt. 22 grs. (10) 125 gals.; 175 gals. 2 qts. (11) 2422 acres 0 roods 12 poles; 705 acres 14 poles; 454 acres 34 poles. (12) 474 leagues 1 mile 6 fur. 6 poles 1 yd.; 432 miles 7 fur. 12 poles. (13) 87 yds. 3 nails; 2298 yds. 3 qrs. 2 nails. (14) 213 weeks 5 days 15 hours; 1740 weeks 18 hours 54 min. (15) 1378 qrs. 5 bush. 2 pecks; 3982 qrs. 7 bush.; 1608 qrs. 3 bush. 3 pecks. (16) £418 19s. 7d.; £867 17s. 8½d.; £3232 2s. 6d. (17) 9696 acres 34 poles; 10442 acres 12 poles; 20138 acres 1 rood 6 poles. (18) £274 4s. 4½d.

(19) £1456 13s. 4d. (20) 598 qrs. 6 bush. 6 qts. (21) 13s. 9d. (22) £11 16s. 6d., and £4 13s. 6d. (23) £39 19s. 3d. (24) £69 7s. 6d. (25) £8 4s. 6d., and £2 12s. 9d. (26) 70 acres 2 roods 29 poles. (27) $190\frac{1}{4}$ lbs. of tea and $66\frac{1}{4}$ lbs. of sugar. (28) 1 hour 27 min. 2 sec. (29) 2 gals. 3 qts. $\frac{1}{2}$ pint. (30) 279 yds. 2 qrs. 3 nails.

XXI.—(1) 1182600000 ; 334416720000000. (2) 2499000 ; 386640000 ; 885530000. (3) 11412162000 ; 153294000 ; 14916000.

XXII.—(1) 474232 ; 397592 ; 72744. (2) 724815 ; 993552 ; 96831296 ; 69326. (3) 6714160 ; 3578126 ; 56179368 ; 934974. (4) 77824 ; 1855365 ; 874259 ; 562770. (5) 1102577 ; 9580977 ; 16527214011. (6) 128583934 ; 3994912 ; 284696426. (7) 7826616 ; 35466730; 70203285. (8) 91554381 ; 1196020822 ; 3705138048. (9) 1737214710 ; 250771828 ; 163543296. (10) 7180927680 ; 253493422 ; 171386558.

XXIII.—(1) 801504 ; 53977968 ; 43785. (2) 710838 ; 6394903263 ; 1718188164. (3) 541904185 ; 7225218 ; 705035. (4) 235246 ; 5045832 ; 3725362746. (5) 379848 ; 4543752 ; 2905955 ; 16566176.

XXIV.—(1) 7801677 ; 6975330 ; 391896702. (2) 4196367 ; 525408 ; 680016. (3) 263334396 ; 5151696 ; 687160. (4) 26606496 ; 255388842 ; 260609202.

XXV.—(1) 27093 ; 54186. (2) 25292 ; 101168. (3) $68\frac{1}{4}$; 82 and 2d. (4) 380 ; 2850. (5) 1250 ; 30000. (6) 9676800 ; 1468800 ; 27900 ; 309600 seconds. (7) 12544 oz. ; 7168 lbs. ; 976 drs. ; 2664 grs. (8) 21120 ft. ; 636 in. ; 174240 sq. ft. ; 9292800 sq. yds. (9) 112 half-pints ; 608 pecks ; 2016 gills ; 5184 pints. (10) 259200. (11) 2000.

XXVI.—(1) $14592\frac{3}{8}$; 105907 ; $11306\frac{10}{11}$; $1366\frac{7}{8}$; $49748\frac{1}{11}$; 121639 ; $54\frac{1}{5}$. (2) $10425\frac{6}{7}$; $7904\frac{1}{11}$; $33635\frac{3}{8}$; 5902 ; 16583 ; $8229\frac{1}{4}$. (3) 10972 ; 120247 ; $60765\frac{5}{12}$; $6548\frac{1}{6}$; $9121\frac{1}{2}$. (4) $56\frac{3}{8}$; 28 ; 462 ; $32\frac{8}{11}$. (5) 16692. (6) 6237. (7) 36753. (8) 3968.

XXVII.—(1) $155\frac{23}{28}$; $277\frac{10}{43}$; $11462\frac{84}{85}$. (2) $570\frac{31}{49}$; $3153\frac{62}{129}$; $332\frac{14}{75}$. (3) $26\frac{8}{21}$; $1428\frac{1}{44}$; $201\frac{59}{68}$; $330\frac{24}{25}$. (4) $2201\frac{10}{13}$; $741\frac{6}{37}$; $972\frac{19}{59}$. (5) $1970\frac{2}{47}$; $630\frac{28}{65}$; $1467\frac{19}{56}$. (6) $867\frac{82}{85}$; $827\frac{25}{48}$; $275\frac{14}{103}$. (7) 255. (8) £1488. (9) £1104. (10) 99. (11) 5931.

XXVIII.—(1) $551\frac{327}{607}$; $24987\frac{203}{867}$; $2024\frac{10}{39}$. (2) $8943\frac{31}{67}$; $2212\frac{187}{327}$;

$46006\frac{87}{126}$. (3) $52548\frac{21}{78}$; $13833\frac{41}{236}$; $18960\frac{354}{427}$. (4) $965\frac{272}{526}$; $46094\frac{12}{18}$. (5) $1120\frac{43}{65}$; $16047\frac{18}{49}$; $12\frac{3610}{4098}$. (6) $6528\frac{29}{47}$; $170\frac{129}{163}$; $491\frac{28}{87}$. (7) $572\frac{176}{541}$; $694\frac{489}{587}$; $58\frac{571}{675}$. (8) $52\frac{1}{49}$; $23191\frac{4}{29}$; $3726\frac{29}{123}$. (9) $110\frac{11}{62}$; $529\frac{183}{526}$; $647\frac{67}{824}$. (10) $963\frac{11}{31}$; $18813\frac{17}{27}$; $7\frac{2}{61}$; $1389\frac{28}{46}$. (11) $982\frac{4}{17}$; $63\frac{240}{263}$; $147\frac{60}{81}$. (12) $17\frac{23}{44}$; $69\frac{8}{21}$. (13) 15361. (14) 4044. (15) 56384.

XXIX.—(1) £180 17s. 11d.; £8 15s. $0\frac{3}{4}$d.; £97 16s. $10\frac{3}{4}$d. (2) £59 11s. 3d.; £45 15s. $7\frac{1}{4}$d. + $\frac{2}{3}$; £142 17s. $1\frac{1}{2}$d. + $\frac{6}{7}$. (3) £896 7s. $11\frac{1}{2}$d.; £784 6s. $11\frac{1}{2}$d. + $\frac{1}{4}$; £697 3s. $11\frac{1}{2}$d. + $\frac{4}{5}$. (4) £10 12s. $7\frac{3}{4}$d. + $\frac{9}{28}$; £10 7s. $7\frac{1}{2}$d. + $\frac{332}{352}$. (5) £2 9s. 5d. + $\frac{41}{614}$; £91 15s. $5\frac{1}{2}$d. + $\frac{10}{17}$; £5 7s. $5\frac{1}{4}$d. + $\frac{479}{510}$. (6) £213 1s. $2\frac{3}{4}$d.; £23 4s. $11\frac{3}{4}$d. + $\frac{67}{187}$; £44 12s. $6\frac{1}{2}$d. + $\frac{34}{45}$. (7) £47 5s. $10\frac{153}{331}$d.; £955 10s. 5d. + $\frac{8}{65}$; £2 $0\frac{3}{4}$d. + $\frac{76}{185}$. (8) 4 cwt. 3 qrs. 16 lbs. 10 oz. $10\frac{10}{13}$ drs.; 14 cwt. 2 qrs. 20 lbs. 4 oz. $6\frac{18}{29}$ drs.; 1 cwt. 1 qr. 21 lbs. 12 oz. $12\frac{4}{5}$ drs. (9) 9 oz. 8 dwt.; 1 lb. 1 oz. 12 dwt. $21\frac{1}{14}$ grs.; 2 oz. 15 dwt. $17\frac{1}{125}$ grs. (10) 1 mile 26 poles 1 ft. $1\frac{2}{15}$ barleycorns; 8 miles 3 fur. 1 pole 4 yds. 2 in. $1\frac{1}{23}$ barleycorns; 21 poles 1 yd. 2 ft. $4\frac{4}{11}$in. (11) 4 acres 1 rood; 2 acres 3 roods 13 poles 10 yds. 108 in.; 1 acre 2 roods 7 poles 8 yds. 2 ft. 36 in.; 1 acre 1 rood 9 poles 6 yds. 8 ft. $119\frac{1}{13}$ in. (12) 46 qrs. 4 bush. $1\frac{1}{8}$ pecks; 5 bush. $1\frac{8}{27}$ pecks; 1 qr. 5 bush. 1 peck $1\frac{1}{3}$ gal.; £78 16s. $4\frac{1}{4}$d.

XXX.—(1) 1100; $7661\frac{105}{121}$; $2790\frac{13}{30}$. (2) $4893\frac{58}{77}$; 1000; $13890\frac{14}{15}$. (3) $470\frac{265}{524}$; $3274\frac{47}{138}$; $727\frac{35}{41}$. (4) $153\frac{3}{40}$; $115\frac{183}{278}$; $8168\frac{8}{45}$. (5) $1057\frac{28}{29}$; 6270; $127\frac{18}{1040}$. (6) $2830\frac{750}{1023}$; $59\frac{30}{73}$. (7) $32\frac{85}{87}$; $898\frac{494}{920}$; $120\frac{18}{35}$. (8) 224; $19033\frac{3}{5}$; $347\frac{80}{87}$. (9) $2181\frac{9}{11}$. (10) $1732\frac{1}{2}$ fourpenny pieces, 1155 sixpences. (11) $7556\frac{4}{21}$. (12) $3310\frac{10}{21}$ half-guineas; £$912\frac{9}{20}$. (13) $107\frac{7}{11}$. (14) $175\frac{10}{13}$. (15) 7040. (16) $2174\frac{362}{487}$. (17) 5 hours 16 min. (18) $1404\frac{26}{41}$. (19) $441\frac{9}{119}$. (20) $51723\frac{48}{149}$.

XXXVI.—(1) £1 5s. $8\frac{1}{4}$d.; £358 3s. $5\frac{1}{4}$d. (2) £29 2s. $6\frac{3}{4}$d.; £58 5s. $1\frac{1}{2}$d.; £116 10s. 3d. (3) 6 hours 37 min. 36 sec.; 110 days 9 hours 20 min.; 1145 years 13 weeks 4 days. (4) 2432 crowns $10\frac{1}{4}$d. (5) 51 miles 2 fur. 1 pole 3 yds. $2\frac{1}{2}$ft.; 23 yds. 2 ft. 11 in. (6) 12 tons 9 cwt. 3 qrs. 17 lbs. 2 oz. (7) 1092. (8) $1\frac{1388}{5760}$ troy, $1\frac{98}{7000}$ avoirdupois. (9) 9 acres 2 roods $17\frac{648}{1089}$ poles. (10) 10 acres 1 rood 33 poles 12 yds. 5 ft. 108 in. (11) $123\frac{23}{24}$. (12) $143\frac{13}{20}$. (13) $19013\frac{1}{4}$ years. (14) 93 tons 15 cwt. 2 qrs. 12 lbs. 13 oz. 4 drs. (15) 131. (16) 112 lbs. 6 oz. 16 dwts. 1 gr.

XXXVII.—(1) 5 × 3; 16 × 4; 3 × 7. (2) $\frac{18}{2}$; 8 × 4; 17 × 10. (3) $\frac{b}{3}$; ay; $\frac{q}{n}$.

XXXVIII.—(1) 48; 98; 2. (2) 9; 3; 63. (3) 1; 288; 162. (4) $9a$; nnx; $\frac{p}{n}$.

XXXIX.—(1) 64; $\frac{4}{8}$. (2) 27; $49\frac{353}{391}$. (3) $1575\frac{1}{3}$; $447\frac{555}{716}$. (4) $366\frac{74}{157}$; $71\frac{714}{812}$. (5) $21058\frac{88}{128}$; $6\frac{2}{3}$. (6) 12; $\frac{ab}{c}$.

XL.—(1) 21. (2) 64. (3) 30. (4) 10. (5) 34. (6) $\frac{b}{3}$. (7) 15. (8) $4y$. (9) qm. (10) $\frac{b}{n}$.

XLIII.—(1) $83\frac{5}{7}$; $15\frac{3}{13}$; $792\frac{18}{50}$. (2) $8\frac{3}{4}$; $6461\frac{2}{5}$; $414\frac{8}{9}$. (3) 72; $866\frac{2}{3}$; $21\frac{6}{27}$.

XLIV.—(1) $185\frac{1}{5}$; $150\frac{4}{11}$; $118\frac{6}{17}$. (2) $135\frac{6}{13}$; $122\frac{5}{8}$; $137\frac{5}{23}$. (3) $3070\frac{2}{3}$; $1149\frac{3}{5}$; $636\frac{4}{25}$. (4) $3573\frac{1}{7}$; 198; $212\frac{5}{23}$. (5) $317\frac{5}{22}$; $1117\frac{1}{11}$; $1943\frac{7}{21}$. (6) $8271\frac{47}{51}$; $413\frac{19}{37}$; $617\frac{11}{37}$.

XLV.—(1) $\frac{2}{5}$; 4; $4\frac{1}{3}$. (2) $7\frac{25}{27}$; $51\frac{3}{4}$. (3) $3\frac{1}{2}$; 6; $\frac{d}{ax}$. (4) $\frac{12}{125}$. (5) 5.

XLVI.—(1) 140535. (2) $56\frac{16}{18}$; $41\frac{259}{595}$; $1327\frac{1}{3}$. (3) 6s. $5\frac{1}{2}$d. + $\frac{1}{3}$. (4) £25 1s. 8d.; £32 16s. 3d. (5) £19 4s. 9d. (6) 3s. $3\frac{1}{4}$d. $\frac{11}{28}$; £2 18s. $9\frac{3}{4}$d. $\frac{9}{17}$. (7) 622640. (8) 448416. (9) (*a*) 80; (*b*) 940; (*c*) 175360; 2577763; (*d*) 2668; (*e*) 257041; $312\frac{5}{17}$. (10) $10195\frac{75}{87}$. (11) 37128. (12) 525948. (13) 1407. (14) 4528. (15) Belgium 320 francs 67 centimes; Prussia 84 thalers 25 s. g.; Frankfort 152 florins 42 kreutzers. (16) 118839. (17) 96. (18) 976. (19) $380\frac{4}{17}$. (20) 1 acre 3 roods 21 poles 4 yds. 4 ft. 48 in. (21) 11 lbs. 9 oz. 19 dwt. 12 grs. (22) £819 12s. $8\frac{1}{4}$d. (23) 2 lbs. 4 oz. 9 dwt. $1\frac{1}{13}$ grs. (24) 5 lbs. 6 dwt. $22\frac{1}{2}$ grs. (25) 9 miles 1 fur. 25 poles 2 yds. 1 ft. 6 in. (26) 139920. (27) £29 4s. $4\frac{1}{2}$d. (28) 3 ft. $6\frac{2}{3}$ in. (29) 25637 + $\frac{3}{4}$d. (30) £1726 13s. 4d. (31) 1s. $1\frac{1}{2}$d. (32) £281 5s. (33) 1st = £5333 6s. 8d.; 2nd = £8000; 3rd = £10666 13s. 4d. (34) 537. (35) 4 miles 0 fur. 31 poles 4 yds. 1 ft. $7\frac{29}{31}$ in. (36) £1 5s. $9\frac{3}{73}$d. (37) 49116. (38) 623 oz. 18 dwt. 18 grs. (39) $522\frac{199}{235}$. (40) 1353 yds. 2 qrs. 2 nails. (41) £5989 13s. 4d.

(42) 115794. (43) 528093440. (44) 20475. (45) £66 7s. 5d. (46) £11660. (47) £6 17s. $7\frac{1}{2}$d. (48) 625. (49) $313\frac{1}{3}$ gals. (50) $34\frac{2}{7}$. (51) $179\frac{105}{132}$. (52) 8d. $\frac{384}{420}$. (53) £2 3s. $4\frac{1}{2}$d. $\frac{18}{63}$. (54) 13. (55) 352 years 4 months 1 week $1\frac{200}{243}$ days. (56) £3 15s. 3d. (57) 8326. (58) £1250. (59) $3\frac{17}{36}$ miles.

XLVIII.—(1) 5 ; 36 ; 30. (2) 7 ; 11 ; 5. (3) 8 ; 9 ; 12. (4) 5 ; 4 ; 1. (5) 1 ; 1 ; 4. (6) 2 ; 17 ; 1. (7) 27 ; 27 ; 4. (8) 38 ; 32 ; 132. (9) 1953 ; 618 ; 1321. (10) 2943 ; 86 ; 389.

XLIX.—(1) 25 ; 8 ; 8. (2) 23 ; 2 ; 2. (3) 7 ; 12 ; 3. (4) 12 ; 8. (5) 3 ; 4. (6) 2 ; 5. (7) 7 ; 9.

L.—(1) 347 ; 2 × 29 ; 7 × 7 × 2 × 2. (2) 7 × 13 × 79 ; 31 × 211 ; 4127. (3) 13 × 79 ; 2 × 41 × 43 ; 2 × 3 × 683. (4) 3 × 7723 ; 3 × 3 × 3 × 7 × 29 ; 2 × 35543. (5) 3851 ; 2 × 3 × 11 × 31 ; 2 × 3 × 7 × 109. (6) $3^2 \times 241$; $3^3 \times 2^2 \times 5$; $2 \times 5 \times 7 \times 11$. (7) 2^{12} ; $5^2 \times 3^2$; $2^2 \times 3^3 \times 17$. (8) 2×1531 ; $2^4 \times 3^2 \times 7$; $2^3 \times 1019$.

LII.—(1) 175 ; 216 ; 693. (2) 22734 ; 2100 ; 300. (3) 3936 ; 3248088 ; 1488. (4) 9165 ; 496 ; 150.

LIII.—(1) 150 ; 924 ; 2320. (2) 5040 ; 270. (3) 39270 ; 9500652. (4) 2520 ; 180. (5) 4149360 ; 14364. (6) 25256351428 ; 441000. (7) 324 ; 1404480. (8) 76692 ; 2383920. (9) 533484 ; 129038. (10) 728728 ; 7770. (11) 21945 ; 559062.

LIV.—(1) $\frac{89}{12}$; $\frac{76}{9}$; $\frac{40}{11}$. (2) $\frac{221}{7}$; $\frac{817}{15}$; $\frac{451}{19}$. (3) $\frac{1111}{9}$; $\frac{276}{13}$; $\frac{13}{7}$. (4) $\frac{877}{8}$; $\frac{42}{19}$; $\frac{4954}{27}$. (5) $\frac{150}{7}$; $\frac{189}{9}$; $\frac{mn + a}{n}$. (6) $\frac{128}{9}$; $\frac{811}{11}$; $\frac{ab + 1}{b}$.

LV.—(1) $4\frac{5}{9}$; $7\frac{3}{5}$; $18\frac{1}{15}$. (2) $6\frac{1}{3}$; $37\frac{5}{11}$; $25\frac{64}{125}$. (3) $118\frac{3}{4}$; $12\frac{1}{6}$; $124\frac{4}{5}$; (4) $123\frac{14}{17}$; $17\frac{50}{109}$; $22\frac{58}{187}$. (5) $15\frac{258}{263}$; $17\frac{10}{11}$; $22\frac{7}{18}$. (6) $90\frac{2}{27}$; $4\frac{4}{25}$; $53\frac{28}{41}$.

LVI.—a. (1) $\frac{16}{7}$; $\frac{2}{3}$; $\frac{7}{3}$. (2) $\frac{24}{7}$; $\frac{21}{19}$; 4. (3) $\frac{28}{19}$; 3 ; $\frac{47}{25}$. (4) $\frac{16}{5}$; 4 ; $\frac{221}{54}$. (5) $\frac{14}{6}$; $\frac{9}{2}$; $\frac{14}{3}$. (6) $\frac{85}{19}$; 14 ; $\frac{689}{97}$. b. (1) $\frac{7}{90}$; $\frac{3}{70}$; $\frac{2}{17}$. (2) $\frac{7}{36}$; $\frac{3}{29}$; $\frac{41}{640}$. (3) $\frac{3}{19}$; $\frac{2}{47}$; $\frac{47}{270}$. (4) $\frac{1}{23}$; $\frac{2}{75}$; $\frac{16}{133}$. (5) $\frac{1}{75}$; $\frac{4}{57}$; $\frac{2}{41}$. (6) $\frac{2}{147}$; $\frac{23}{1000}$; $\frac{47}{590}$.

LVII.—b. (1) $\frac{51}{3}$; $\frac{136}{8}$; $\frac{119}{7}$. (2) $\frac{20}{5}$; $\frac{36}{9}$; $\frac{24}{6}$. (3) $\frac{96}{8}$; $\frac{36}{3}$; $\frac{48}{4}$. (4) $\frac{60}{15}$; $\frac{48}{12}$; $\frac{36}{9}$. (5) $\frac{189}{21}$; $\frac{27}{3}$; $\frac{126}{14}$. (6) $\frac{660}{44}$, $\frac{1740}{116}$, $\frac{36540}{2436}$. (7) $\frac{57600}{480}$, $\frac{79920}{666}$, $\frac{68160}{568}$. (8) $\frac{9a}{9}$, $\frac{13a}{13}$, $\frac{15a}{15}$. (9) $\frac{7a}{a}$, $\frac{7b}{b}$ $\frac{7c}{c}$. (10) $\frac{ab}{b}$, $\frac{ac}{c}$, $\frac{ad}{d}$.

LVIII.—(1) $\frac{35}{48}$; $\frac{28}{753}$; $\frac{3}{4}$. (2) $\frac{3}{11}$; $\frac{4}{9}$; $\frac{100}{793}$. (3) $\frac{205}{374}$; $\frac{17}{225}$; $\frac{112}{113}$.

(4) $\frac{55}{114}$; $\frac{74}{85}$; $\frac{251}{264}$. (5) $\frac{4}{7}$; $\frac{3}{80}$; $\frac{7}{18}$. (6) $\frac{59}{1312}$; $\frac{227}{3465}$; $\frac{4}{15}$. (7) $\frac{4}{7}$, $\frac{16}{23}$, $\frac{118}{595}$. (8) $\frac{518}{793}$, $\frac{65}{83}$, $\frac{2030}{5071}$. (9) $\frac{2}{3}$, $\frac{11}{18}$, $\frac{7}{11}$. (10) $\frac{3}{19}$, $\frac{17}{19}$, $\frac{9687}{12763}$.

LIX.—(1) $\frac{80}{280}$; $\frac{105}{280}$; $\frac{224}{280}$. (2) $\frac{1470}{2520}$; $\frac{504}{2520}$; $\frac{180}{2520}$. (3) $\frac{1700}{6460}$; $\frac{2280}{6460}$; $\frac{969}{6460}$. (4) $\frac{8704}{15232}$; $\frac{14280}{15232}$; $\frac{5712}{15232}$; $\frac{3584}{15232}$. (5) $\frac{4368}{7488}$; $\frac{1728}{7488}$; $\frac{2340}{7488}$; $\frac{4992}{7488}$. (6) $\frac{102600}{378200}$; $\frac{12540}{378200}$; $\frac{59400}{378200}$; $\frac{175560}{378200}$. (7) $\frac{504}{3780}$; $\frac{810}{3780}$; $\frac{1680}{3780}$. (8) $\frac{1615}{5168}$; $\frac{1088}{5168}$; $\frac{912}{5168}$. (9) $\frac{5852}{7524}$; $\frac{6732}{7524}$; $\frac{7182}{7524}$. (10) $\frac{518700}{678300}$; $\frac{387600}{678300}$; $\frac{20349}{678300}$; $\frac{214200}{678300}$. (11) $\frac{31360}{67200}$; $\frac{26880}{67200}$; $\frac{12600}{67200}$; $\frac{24000}{67200}$. (12) $\frac{54000}{306000}$; $\frac{102000}{306000}$; $\frac{265200}{306000}$; $\frac{24480}{306000}$.

LX.—(1) $\frac{6}{30}$; $\frac{12}{30}$. (2) $\frac{21}{36}$; $\frac{8}{36}$. (3) $\frac{126}{420}$; $\frac{84}{420}$; $\frac{240}{420}$. (4) $\frac{140}{252}$; $\frac{36}{252}$; $\frac{147}{252}$. (5) $\frac{80}{360}$; $\frac{90}{360}$; $\frac{288}{360}$. (6) $\frac{20}{120}$; $\frac{36}{120}$; $\frac{30}{120}$. (7) $\frac{70}{525}$; $\frac{21}{525}$; $\frac{475}{525}$. (8) $\frac{32}{176}$; $\frac{33}{176}$; $\frac{56}{176}$. (9) $\frac{288}{1008}$; $\frac{168}{1008}$; $\frac{882}{1008}$; $\frac{432}{1008}$. (10) $\frac{20}{120}$; $\frac{70}{120}$; $\frac{56}{120}$; $\frac{18}{120}$. (11) $\frac{108}{1800}$; $\frac{300}{1800}$; $\frac{700}{1800}$; $\frac{228}{1800}$. (12) $\frac{900}{2100}$; $\frac{600}{2100}$; $\frac{12}{2100}$. (13) $\frac{770}{1155}$; $\frac{924}{1155}$; $\frac{990}{1155}$; $\frac{525}{1155}$. (14) $\frac{9}{12}$; $\frac{8}{12}$; $\frac{10}{12}$; $\frac{7}{12}$. (15) $\frac{72}{504}$; $\frac{63}{504}$; $\frac{56}{504}$.

LXI.—(1) $\frac{34}{35}$. (2) $\frac{167}{204}$. (3) $1\frac{142}{399}$. (4) $6\frac{23}{60}$. (5) $23\frac{88}{117}$. (6) $1\frac{589}{1540}$. (7) $10\frac{948}{1015}$. (8) $7\frac{9}{20}$. (9) $6\frac{48}{60}$. (10) $1\frac{1539}{10000}$. (11) $\frac{2503}{2805}$. (12) $5\frac{17}{60}$. (13) $99\frac{65}{84}$. (14) $17\frac{339}{552}$. (15) $102\frac{9}{665}$. (16) $13\frac{17}{20}$. (17) $22\frac{1028}{1953}$. (18) $17\frac{37}{60}$. (19) The whole. (20) $4795\frac{94}{3289}$.

LXII.—(1) $\frac{13}{60}$. (2) $\frac{7}{12}$. (3) $\frac{24}{133}$. (4) $\frac{5}{12}$. (5) $\frac{1}{6}$. (6) $1\frac{109}{228}$. (7) $3\frac{37}{45}$. (8) $1\frac{113}{406}$. (9) $6\frac{11}{20}$. (10) $10\frac{26}{35}$. (11) $7\frac{31}{65}$. (12) $8\frac{577}{1236}$. (13) $147\frac{7}{8}$. (14) $14\frac{1}{45}$. (15) $6\frac{9}{112}$. (16) Sum $1\frac{1}{2}$, diff. $\frac{1}{6}$. (17) $16\frac{59}{60}$. (18) The latter exceeds by $72\frac{5087}{5544}$. (19) The latter by $1\frac{2477}{8778}$. (20) £$999\frac{29}{48}$. (21) $\frac{15}{56}$. (22) £$164\frac{7}{16}$. (23) £$13\frac{11}{16}$. (24) £$3928\frac{4}{7}$. (25) $\frac{852}{1071}$.

LXIII.—(1) $\frac{8}{35}$; $\frac{12}{35}$; $3\frac{4}{7}$. (2) $\frac{12}{25}$; $6\frac{1}{8}$; $18\frac{23}{45}$. (3) $\frac{15}{616}$; $\frac{5}{72}$; $\frac{2}{7}$. (4) $\frac{9}{28}$; $7\frac{1}{2}$; $7\frac{23}{35}$. (5) $\frac{212}{945}$; $\frac{1367}{29070}$. (6) $6\frac{13}{42}$; $\frac{929}{5355}$. (7) $25466\frac{6}{35}$; $6526\frac{127}{165}$; $21\frac{285}{391}$. (8) $21\frac{3}{7}$; $5\frac{5}{8}$; $\frac{63}{200}$. (9) $\frac{1036}{27225}$. (10) $63\frac{86}{153}$. (11) $1\frac{1861}{14560}$. (12) $361\frac{1}{8}$. (13) $236\frac{41}{80}$; $118\frac{29}{80}$. (14) $4\frac{907}{2432}$. (15) $\frac{33}{152}$. (16) $7\frac{71}{80}$ acres. (17) £$11\frac{37}{240}$.

LXIV.—(1) $\frac{1}{7}$; 9 ; $\frac{15}{14}$. (2) $\frac{5}{18}$; $\frac{6}{17}$; $\frac{11}{169}$. (3) $\frac{9}{190}$; $\frac{7}{8}$; $\frac{6}{649}$.

LXV.—(1) $\frac{18}{35}$; 2 ; $1\frac{11}{27}$. (2) $1\frac{19}{56}$; $43\frac{21}{25}$; $6\frac{33}{85}$. (3) $\frac{9}{14}$; $2\frac{6}{17}$; 5 ; $83\frac{359}{380}$. (4) $4\frac{107}{208}$; $74\frac{1}{4}$; $18\frac{381}{413}$ (5) $1\frac{173}{252}$; $106\frac{53}{819}$. (6) $\frac{12}{59}$. (7) $2\frac{38}{49}$. (8) $24\frac{7}{13}$. (9) $\frac{238}{675}$. (10) $\frac{20728}{61430}$. (11) $\frac{17178}{190015}$. (12) $8\frac{1277}{1376}$. (13) $2\frac{18}{45}$; $1\frac{11}{45}$; 1 ; $\frac{25}{81}$. (14) $1\frac{73}{152}$. (15) £$10266\frac{2}{3}$.

LXVI.—(1) $\frac{4}{2555}$ and $\frac{1}{49}$. (2) $\frac{5}{316}$; $\frac{5}{72}$; $22\frac{2}{9}$; $122\frac{2}{3}$; $366\frac{2}{3}$

(3) $\frac{2}{35}$ of £1; $\frac{4}{35}$ of a half-sovereign; $\frac{8}{35}$ of 5s.; $\frac{16}{35}$ of 2s. 6d.; $\frac{8}{7}$ of 1s.; $\frac{16}{7}$ of 6d.; $\frac{24}{7}$ of 4d.; $\frac{32}{7}$ of 3d.; $\frac{96}{7}$ of 1d.; $\frac{192}{7}$ of $\frac{1}{2}$d.; $\frac{384}{7}$ of $\frac{1}{4}$d. (4) $\frac{1}{2304}$ and $\frac{12100}{9}$. (5) $\frac{1}{2176}$ of cwt.; $\frac{7}{136}$ of lb.; $\frac{1225}{19584}$ of lb. Troy; $\frac{1225}{1632}$ of oz.; $\frac{6125}{17}$ of grain. (6) $\frac{5}{2772}$. (7) $\frac{35200}{73}$. (8) $\frac{82}{105}$ of 15s.; $\frac{41}{70}$ of £1; $\frac{41}{350}$ of £5. (9) $\frac{480}{13}$ of a minute; $\frac{1}{39}$ of a day; $\frac{1}{273}$ of a week. (10) $\frac{144}{25}$ of a pint; $\frac{3}{350}$ of a puncheon; $\frac{1}{175}$ of a pipe. (11) $\frac{9}{1120}$. (12) $\frac{1}{1632}$. (13) $\frac{19}{32}$ of an acre; $\frac{19}{20480}$ of a mile. (14) $\frac{1}{7500}$. (15) $\frac{56}{3}$ and $\frac{7}{1350}$. (16) $\frac{725}{104632}$ (17) $\frac{3}{6160}$. (18) $\frac{2360591}{315558161}$.

LXVII.—(1) $\frac{5}{2}, \frac{7}{3}, \frac{12}{5}, \frac{43}{18}$; $\frac{1}{3}, \frac{9}{28}, \frac{10}{31}$. (2) $\frac{5}{4}, \frac{58}{45}, \frac{117}{94}, \frac{290}{233}, \frac{407}{327}$; $\frac{6}{7}, \frac{13}{15}, \frac{32}{37}$. (3) $\frac{1}{4}, \frac{25}{101}$; $\frac{9}{2}, \frac{13}{3}, \frac{35}{8}, \frac{83}{19}$. (4) $\frac{1}{6}, \frac{1}{7}, \frac{4}{27}, \frac{21}{142}, \frac{46}{311}$; $\frac{4}{1}, \frac{15}{4}, \frac{49}{18}$. (5) $\frac{1}{22}, \frac{2}{45}, \frac{7}{157}, \frac{93}{2086}, \frac{193}{4329}$; $\frac{1}{199}, \frac{83}{16518}, \frac{84}{16717}$. (6) $\frac{1}{2}, \frac{42}{83}$; $\frac{6}{1}, \frac{17}{3}, \frac{23}{4}, \frac{224}{39}, \frac{247}{43}$. (7) $\frac{14}{13}, \frac{29}{27}, \frac{72}{67}, \frac{173}{161}, \frac{418}{389}$; $\frac{6}{5}, \frac{7}{6}, \frac{55}{47}, \frac{62}{53}$. (8) $\frac{2}{3}, \frac{11}{16}, \frac{3}{7}, \frac{7}{16}, \frac{17}{39}, \frac{24}{55}$. (9) $\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{4}{11}$; $\frac{1}{3}, \frac{22}{67}, \frac{23}{70}, \frac{114}{347}$. (10) $\frac{1}{3}, \frac{7}{22}, \frac{8}{25}, \frac{23}{72}, \frac{100}{313}, \frac{523}{1637}, \frac{623}{1950}$; $\frac{1}{20}, \frac{2}{41}, \frac{15}{307}, \frac{77}{1576}, \frac{169}{3459}, \frac{246}{5035}$. (11) $\frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{7}{12}, \frac{11}{19}, \frac{40}{69}, \frac{91}{157}, \frac{131}{226}, \frac{222}{383}, \frac{575}{992}$; $\frac{1}{3}, \frac{3}{10}, \frac{13}{43}, \frac{29}{96}, \frac{100}{331}, \frac{129}{427}, \frac{229}{758}$. (12) $\frac{1}{27}, \frac{4}{109}, \frac{153}{4169}, \frac{1228}{33461}$; $\frac{1}{14}, \frac{10}{141}, \frac{11}{155}, \frac{32}{451}, \frac{43}{606}, \frac{161}{2269}, \frac{529}{7413}, \frac{1738}{24508}, \frac{5748}{80937}$.

LXVIII.—(1) 2s. $3\frac{9}{16}$d., and £3 10s. (2) 2. qrs. 24 lbs., and 5 cwt. 1 qr. $9\frac{1}{3}$ lbs. (3) 56, and $2\frac{19}{29}$ miles. (4) £13 16s. $3\frac{15}{19}$d., and £4 19s. $8\frac{12}{23}$d. (5) £3 13s. $3\frac{3}{8}$d., and £10 13s. $10\frac{2}{3}$d. (6) £68 8s. $8\frac{8}{19}$d., and £12 18s. $10\frac{2}{7}$d. (7) £9 16s., and £27 17s. $9\frac{48}{49}$d. (8) 5 hhds. 1 kil. 6 gals., and 17 hhds. 1 runlet $7\frac{1}{5}$ gals. (9) 1 mile 1 fur. 5 poles 3 yds. 2 ft. $9\frac{3}{7}$ in., and 3 roods 18 poles 20 yds. 1 ft. 72 in., and 373 acres 1 rood 13 poles 10 yds. 108 in. (10) 5 bush. 3 pecks 1 gal.; 1 bush. 3 pecks $1\frac{1}{17}$ gals.; 1 bush. 2 pecks 2 quarts $1\frac{5}{7}$ pints. (11) £1 4s. $\frac{96}{133}$d., and £1 13s. $2\frac{26}{53}$d., and $3\frac{17}{21}$d. (12) 2 weeks 6 days, and 8 weeks $\frac{2}{13}$ of a day, or $56\frac{2}{13}$ days. (13) 3 miles 27 poles 2 yds. $1\frac{1}{14}$ ft., and 5 fur. 25 poles 4 yds. $2\frac{19}{34}$ ft. (14) 7 cwt. 6 lbs. $9\frac{7}{17}$ oz. and 2 qrs. $1\frac{1}{55}$ lbs. (15) 18s. $5\frac{613}{2431}$d. (16) 409 acres. (17) 3s. $11\frac{1}{16}$d. (18) $\frac{3}{3520}$.

LXIX.—(1) 6 hours, and 3 days 12 hours. (2) 4s. 3d., and £13 14s. 6d. (3) 45 yds., and $33\frac{1}{3}$ ft. (4) 1 lb. 3 dwt. 16 grs. (5) 21 lbs. 6 oz. 8 drs., and 6 cwt. 2 qrs. 24 lbs. 14 oz. $3\frac{5}{9}$ drs. (6) $26\frac{1}{15}$, and $7\frac{9}{125}$. (7) £15. (8) 6 acres 1 rood 20 perches. (9) £2 0s. $6\frac{1}{2}$d. (10) $17\frac{1}{2}$ lunar months.

LXX.—(1) $\frac{13}{648}$. (2) $\frac{9}{56}$. (3) $\frac{11}{512}$. (4) $\frac{409}{11200}$. (5) $\frac{1}{360}$ (6) $\frac{2063}{12000}$; $\frac{131}{6000}$. (7) $\frac{17}{288}$. (8) $\frac{161}{240}$; $\frac{23}{144}$. (9) $\frac{13}{21136}$ (10) $\frac{1157309}{59392000000}$. (11) $\frac{119}{3900}$.

LXXI.—(1) £53892 2s. 6d.; £5600 9s. 9¾d. (2) £70331 14s. 8d.; £10854 15s. (3) £130714 4s. 1½d.; £8676 5s. (4) £33024 19s. 7d.; £115372 3s. 10½d. (5) £295542 7s. 3¾d.; £273004 15s. 10½d. (6) £554 18s. 6d.; £16066 10s. 6d. (7) £4384 15s 10¾d., and £63722 10s. 0½d. (8) £604 19s. 6¾d. and £10833 10s. 7½d. (9) £6259 7s. 0½d., and £306390 3s. (10) £13267 0s. 11¼d., and £40690 1s. 8d. (11) £26995 3s. and £16703 18s. 10½d. (12) £2206955 19s. 10¼d., and £35451 18s. 2d. (13) £9343 0s. 0¾d., and £1336 3s. 3½d. (14) £348046 2s. 10d., and £76173 3s. 6d. (15) £55234 1s. 3d, and £1946 11s. 9d. (16) £281 18s. 4½d., and £29958 13s. 1¾d. (17) £327951 6s. 10d., and £17705 3s. 8¼d. (18) £31993 2s. 6¾d., and £2760 3s. 3d. (19) £578708 13s. 4d., and £217975 15s. 4d. (20) £462212 4s. 3d., and £1168918 16s. 9¼d. (21) £420499 2s. 7d. (22) £1051855.

LXXII.—(1) £5 3s. 7½d. (2) £48 0s. 8d. (3) £532 5s. 5¼d. (4) £9 1s. 4¼d. (5) £31 3s. 6¾d. (6) £207 10s. 6½d. (7) £153 10s. 4½d. (8) £461 7s. 10½d. (9) 16s. 9½d. (10) 5s. 6¼d. (11) 5d. (12) 12s. 8¼d. (13) £45 7s. 2¼d. (14) £339 1s. 3¾d. (15) £145 16s. 5¼d. (16) £165 16s. 6¼d. (17) 1105 13s. 6$\frac{7}{20}$d. (18) £1141 18s. 3⅜d. (19) £2670 6s. 6¾d. (20) £3757 6s. 1¾d. (21) 795 fr. 30 cent. (22) £2637 15s. 0¾d.; £2577 13s. 4¾d.

GENERAL EXERCISES ON VULGAR FRACTIONS.

(1) 17s. 4$\frac{49}{66}$d. (2) 37½ days. (3) 13$\frac{941}{8000}$. (4) $\frac{21}{160}$. (5) $\frac{127}{1520}$. (6) 11$\frac{5}{8}$. (7) 2$\frac{13}{40}$; 1$\frac{193}{1235}$. (8) 1$\frac{22}{105}$; 7$\frac{14}{15}$. (9) $\frac{307}{495}$. (10) $\frac{1}{1}+\frac{1}{17}+\frac{1}{4}$. (11) £16 11s. 8$\frac{11}{12}$d. (12) £7 16s. 5$\frac{39}{80}$d. (13) $\frac{4}{13}$. (14) 2$\frac{425}{748}$; 46$\frac{25}{41}$. (15) $\frac{4623}{10472}$. (16) 1913$\frac{137}{280}$. (17) $\frac{1}{6}$. (18) 2$\frac{97}{144}$. (19) 112. (20) 1500. (21) £143 10s. 4d. (22) 42. (23) 3$\frac{127}{165}$; 16$\frac{349}{532}$. (24) £311 6s. 5$\frac{49}{67}$d. (25) $\frac{87}{164}$. (26) 11. (27) 7$\frac{27}{83}$ inches. (28) £1 10s. 4d. (29) $\frac{13}{207}$. (30) 72$\frac{1901}{2464}$. (31) $\frac{638}{1102}$; $\frac{684}{1102}$; $\frac{667}{1102}$. (32) 1s. 0$\frac{12}{13}$d.; 1s. 0$\frac{3}{5}$d.; 1s. 1$\frac{5}{7}$d. (33) 1$\frac{61}{64}$. (34) £884 18s. 0¼d. (35) £91 11s. 10$\frac{2}{49}$d. (36) £180. (37) $\frac{559}{900000}$. (38) 27 roods 16 poles. (39) 18° F. = 8° R. = 10° C., and 14$\frac{7}{16}$° F. = 6$\frac{5}{12}$° R. = 8$\frac{1}{48}$° C.

LXXVII.—(1) ·57142; 1·82746; 1·06896. (2) ·66936; ·8144; ·8. (3) ·58333; ·72; 2. (4) ·921875; ·168; ·88888. (5) 1·0625; ·135; ·15813. (6) ·16; ·8; ·36. (7) 2·875; 13·4; 122 7+. (8) ·52941; ·42490; ·102. (9) ·3125; 12·75; 2·57142. (10) ·17857; ·75789. (11) ·059312; ·86806. (12) ·41379; 26.

LXXVIII.—(1) ·$\dot{6}\dot{3}6363$; ·$958\dot{5}$. (2) ·$\dot{5}7142\dot{8}$; ·$58\dot{3}$; ·$\dot{4}2105263157894736\dot{8}$. (3) ·$\dot{3}8461\dot{5}$; ·$\dot{3}52941176470588\dot{2}$; ·$\dot{2}\dot{6}$. (4) ·$\dot{7}39130434782608695652\dot{1}$; ·$\dot{6}3157894736842105\dot{2}$; ·$\dot{2}\dot{2}$. (5) ·$\dot{4}70588235294117\dot{6}$; ·$\dot{3}8461\dot{5}$; ·$\dot{4}2857\dot{1}$. (6) ·$\dot{6}\dot{6}$; ·$\dot{6}$; ·$\dot{8}$. (7) ·$\dot{4}2857\dot{1}$; ·$0\dot{8}5714\dot{2}$; ·$\dot{8}\dot{1}$. (8) ·$\dot{4}\dot{5}$; ·$\dot{7}$; ·$\dot{1}\dot{8}$. (9) ·$91\dot{6}$; ·$\dot{8}57142$; ·$\dot{8}82352941176470\dot{5}$.

LXXIX.—(1) $\frac{47}{99}$; $\frac{5}{9}$; $\frac{5}{90}$. (2) $\frac{827}{999}$; $\frac{95238}{99999}$; $\frac{714285}{999999}$. (3) $\frac{185}{99900}$; $\frac{8129}{9990}$; $1\frac{6946}{9900}$. (4) $\frac{28144}{99900}$; $3\frac{411}{990}$; $\frac{70168}{99900}$. (5) $6\frac{38}{999}$; $\frac{7054}{9900}$; $\frac{80488}{99900}$. (6) $\frac{13574}{99900}$; $\frac{719}{999}$; $6\frac{27}{9990}$. (7) $1\frac{27289}{99900}$; $45\frac{458}{900}$; $\frac{827}{999}$. (8) $\frac{7025}{9990}$; $6\frac{37}{90}$; $7\frac{43}{990}$. (9) $27\frac{56}{990}$; $3\frac{75}{9900}$; $4\frac{127}{999}$. (10) $11\frac{3}{9}$ $\frac{2}{9}$; $4\frac{152}{900}$; $\frac{50927}{99990}$. (11) $\frac{7}{990}$.

LXXXII.—(1) 587·167. (2) 303·4867. (3) 2064·1614. (4) 150·9036. (5) 815·6894. (6) 235·1545. (7) 915·2858. (8) 1370·876842. (9) 1045·9222. (10) 532·0767. (11) 1265·6408. (12) 12890·28028. (13) ·217047. (14) 24·260934. (15) 5·065874. (16) 16·037018. (17) 11556·09361414.

LXXXIII.—(1) 708·7786; 215·557. (2) 52·3416; 80·263. (3) 2090·35238; 17·82. (4) 149·193; 216·06528. (5) 1158·002; 38·835. (6) ·999; 729·7913. (7) 2123·4922; 62·33155. (8) 2141·181; 41·48357. (9) 871·2199; 277·01714. (10) 28·0862; 319·6. (11) ·7113. (12) ·5016; 68·96. (13) 4·$262\dot{0}\dot{2}$; 72·5847. (14) 27·912967. (15) ·6334. (16) ·33116. (17) ·95. (18) ·$61617\dot{4}285$. (19) Latter by ·$0\dot{3}17\dot{3}$. (20) 7·717. (21) 9·44°.

LXXXIV.—(1) 1776·73; 6531·71. (2) 296·9225; 354·2512. (3) 1221815·868; 83·65427. (4) 8·9332828; ·1360485. (5) 19533·176; 424097·46822. (6) 117467·0923216; 150352·146945. (7) 98·2602; 77·8321752. (8) 379·$117\dot{2}$; 17·303. (9) 4729·78826304; ·2628. (10) 143·507702; 711·59055. (11) 886761·65808; 296.4114. (12) 475·135024. (14) The former by £813·2. (15) 4090·714. (16) £180033·82575. (17) £236·52178. (18) 114·2164.

LXXXV.—(1) 6·$26\dot{3}$; ·24094; 15·097202. (2) 3·0495; ·31008; ·15421. (3) ·83235; ·00603738; 4·98572. (4) 78·1482; 43·0547; ·0246. (5) ·0049876; 112·$7832\dot{4}$; ·01767. (6) ·000139; 1·7453; ·2885428. (7) £53·508571. (8) Each man had ·96375, and paid £1·1515625. (9) 674·$1723\dot{5}$. (10) £17·4188017.

LXXXVI.—(1) 5316·883; 25·$\dot{7}\dot{5}$; 5·8. (2) 2006·99;

1000; 5·3623188. (3) 111·81102; 1·356633; 176·591375. (4) 129·903703; 3448.148; 69523·33664. (5) 20342·85714; 10; 100. (6) 10·73379; 13·031114. (7) 7·5008; 75·0437. (8) 23·20518; ·24058. (9) 20·01796. (10) 4·919807. (11) 2073·34456. (12) 8643·402399.

LXXXVII.—(1) ·01; 17 618; 1·18235294. (2) 47·072; ·01722583; ·2195956. (3) 33·0457339449; 256·592083; 36 34. (4) 2·405283019; 6142·3; 6·96992982. (5) 1·95383; ·63932269; ·147245421. (6) ·968965517; 2·93685039; 6·48354858. (7) 405·7604395; 29·31328671. (8) ·00932808; 18·79425003. (9) 7·91610181; 17·6031584. (10) ·019883481; ·0448478229; 5·1249429. (11) ·15311. (12) 1·24034. (13) 1·35818. (14) 15·063941

LXXXVIII.—(1) 8·2052. (2) ·3. (3) £1 19s. 4½d. (4) 11$\frac{513}{625}$= 11·8208. (5) £1 5s. 8d. (6) ·21.

LXXXIX.—(1) 10·018; 1198·784; 3·65. (2) 341·071; 788·424; 9·129. (3) 572·931; 1366·36; 199·248. (4) 22·142; 43·897; 1·756. (5) 78·83; 13·841. (6) 333·795; 292·045. (7) 132·351; 2·79.

XC.—(1) 1·972; 3·678; 1·819. (2) ·034; 3·27. (3) 2·382. ·033; 6·402. (4) ·01023; 1·302; 62.975. (5) 13·009; ·023; 45·621.

XCI.—(1) ·00925 fur.; ·0015625 of mile; ·0003854·16 of league; 2·035 of yd.; 6·105 of ft.; 73·26 in. (2) £·0375; ·075 of 10s.; ·15 of crown; ·3 of half-crown; ·375 of florin; ·75 of 1s.; 1·5 of 6d.; 2·25 of groat; 3·0 of 3d.; 9·0 of 1d.; 18·0 of halfpenny; 36·0 of farthing. (3) 97·44; ·00005138. (4) ·023 of a qr.; ·736 of a peck. (5) 3·9 of 3d.; ·39 of 2s. 6d., and ·065 of 15s. (6) ·074865 of £10, and 718·704 of a farthing. (7) ·1675 of £1; ·335 of 10s., and 80·4 of halfpenny. (8) 3·5 of a florin. (9) ·000241071 of 11 cwt. (10) ·006203125 bush. (11) 609·6 gr.; 30·48 scr., and 25·4 dwt. (12) ·0001523437 of a square mile.

XCII.—(1) 17 weeks 6 days 9 hours 33 min. 7·2 sec.; 1 pint 1·696 gills; 32 weeks 5 days 5 hours 19 min. 40·8 sec. (2) £2 3s. 6½d.; £5 14s. 6½d.; 16s. 5½d. + ¾. (3) £17 10s. 0½d.; £1 12s. 6¼d.; £4 15s. 7d. (4) 13s. 7d.; £2 9s. 4¾d.; £1 5s. 7d. (5) £103 9s. 7d.; £2 3s. 8½d.; £7 1s. 11d. (6) £7 16s. 10d.; £303 15s. 11d.; £2 1s. 8¾d. (7) 29 cwt. 1 qr. 8 lbs. 4 oz. 9 drs.;

5 cwt. 3 qrs. 13 lbs. 7 oz. ; 2 qrs. 27 lbs. 2 oz. 8 drs. (8) 3s. 6d. ; 19s. 5¼d. ; 2s. 2d. (9) 3 days 6 hours 28 min. 48 sec. ; 4 weeks 5 days 2 hours 18 min. 14 sec. ; 39 hours 16 min. 40 sec. (10) 1 mile 2 fur. 12 poles 4 yds. 1 ft. 2⅞ in. ; 27 fur. 11 poles 1 yd. 3 in. ; 3 leagues 1 fur. 8 poles. (11) 27 yds. 1 ft. 1 in. ; 4 poles 1 yd. 2·8 in. ; 282 miles 5 fur 30 poles 2 yds. 7 in. (11) 37 acres 1 rood ; 48 acres 3 roods 35 poles 15 yds. 6 ft. ; 25 roods 13 poles 18 yds. 1 ft. 10 in. (13) 13 yds. 3 ft. 60 in. ; 14 miles 150 acres 3 roods 16 poles 81 yds. 2 ft. 95 in. ; 3 roods 36 poles 14 yds. 4 ft. 97 in. (14) 17 gals. 3 qts. 2 gills ; 212 gals. 1 qt. 1 gill ; 1 gal. ·56 gills. (15) 29 qrs. 5 bush. 3 pecks 3 qts. ; 2 pecks 1 gal. ; 53 qrs. 1 bush. 3 pecks 2 qts. (16) 23 yds. 1 qr. 2 nails ; 4 yds. 3 qrs. 2 in. ; 8 yds. 1 qr. 1 in.

XCIII.—(1) 17·540625 ; 2·28125 ; 8·7677. (2) 25·93125 ; 1·52395 ; 297·53125. (3) 17·767857 ; ·83035 ; 3·8169. (4) 1179·6071 ; 40·96428 ; 68·3214. (5) 17·375 ; 57 ; 364. (6) ·8281 ; 29·428571 ; 511·25. (7) ·788255. (8) 3·565476190. (9) ·5384645 ; ·741354. (10) ·6 ; ·571428. (11) ·40972 ; 77·5. (12) 19·3125 ; 2·125.

XCIV.—(1) 2·733 ; 3·539 ; 27·228. (2) 7·918 ; 235·762 ; 228·331. (3) 14·887 ; 628·229 ; 5·667. (4) 1·581 ; 274·812 ; 2·04. (5) 2·367 ; 8·561 ; 17·339. (6) 8·118 ; 37·113 ; 11·467.

XCV.—(1) £19 17s. 3½d. ; £25 7s. 5d. ; £176 2s. 8½d. (2) £17 10s. 6¾d. ; £1 1s. 3½d. ; £18 7s. 5¼d. ; £20 19s. 3¾d. ; £8 5s. 5d. (3) £16 14s. 5¼d. ; £4 2s. 6¾d. ; £3 1s. 5¾d. ; £4 2s. 1½d. ; £2 1s. 10½d. (4) £4 13s. 11¼d. ; £18 6s. 6d. ; £71 8s. 5½d. ; £58 7s. 5½d. ; £41 12s. 7d. (5) £7 5s. 8¼d. ; £5 18s. 0¼d. ; [illegible] 12s. 5¾d. ; £5834 14s. 5¼d. ; £209 12s. 11d. (6) 17s. 5½d. ; [illegible] 19s. 2½d. ; 11½d. ; £814 15s. 1½d. ; £203 17s. 5¼d. (7) [illegible] 1s. 8d. ; 9s. 2½d. ; £8 13s. 11d. ; £23 14s. 5d. ; 1s. 5½d.

[illegible]) £17·514583 ; £27·669791 ; £109·427083. (2) [illegible] £12·336458 ; £1·3135416. (3) £28·4677083 ; [illegible]78125. (4) £17·035416 ; £29·8135416 ;

[illegible] ON FRACTIONS.

[illegible]3. (3) 1½ hours. (4) 1. [illegible] min. 11·04 sec. ; £45 9s. 6¾d.

(7) $\cdot 0\dot{5}71428\dot{8}$; $\cdot 00\dot{1}095890\dot{4}$. (8) $\frac{272}{371}$; $\frac{220}{667}$; $1\frac{159}{161}$; $1\frac{97}{400}$; $2\frac{24}{77}$. (9) 6s. $9\frac{1}{7}$d. (10) 3s. $1\frac{1}{4}$d. + $\frac{10}{21}$ of a farthing. (11) 19s. 8d. (12) ·0288 and 125000; ·020412 and $2603\cdot571428\dot{5}$. (13) £1000. (14) $\frac{9311}{12375}$. (15) $6\cdot\dot{7}14285\dot{7}$. (16) £2 13s. $7\frac{1}{4}$d. and £48 2s. $11\frac{1}{2}$d. (17) $\frac{1}{3}$, $\frac{22}{67}$, $\frac{23}{70}$, $\frac{114}{347}$; $3\frac{1}{7}$, $3\frac{15}{106}$, $3\frac{16}{113}$, $3\frac{415}{2931}$, $3\frac{431}{3044}$, $3\frac{3432}{24239}$. (18) £354 10s. $3\frac{3}{4}$d. $\frac{9}{20}$. (19) 240 pints. (20) ·369. (21) 11 and 1. (22) 43·039783 ft., 6·1575164 yds., and 89·03894378 miles. (23) 3·997 in., 1·049 ft., 3·526 miles. (24) £107·3006; £355·3. (25) (a.) £1684·75; £23982·05; £1739·806. (b.) £15945·491; £710·963; £455·91. (c.) £368·756; £8978 13s. 9d.; £222 12s. $7\frac{3}{4}$d. (d.) £115 7s. $4\frac{3}{4}$d.; £97493 15s. $3\frac{3}{4}$d.; £42 14s. $9\frac{3}{4}$d. (26) £728. (27) $13\frac{1}{23}$ min. (28) 2·1714285. (29) $\frac{582}{1263}$; $\frac{671}{1263}$. (30) 6296·41; 9189·504; 27963·907; 1465·55. (31) 3·81024. (32) 25·7069 days. (33) 8 6648; 51·988875; 207·9555; 3050·014. (34) 98 fr. 8 cent.; 461 fr. 81 cent.; 5399·92 fr. (35) 14s. 3d.; £86 13s. 7d.; £42 9s. $0\frac{3}{4}$d. (36) £18 17s. $4\frac{1}{2}$d. (37) 17952 : 29920. (38) 58 a. 1 r. 22 p. 1 yd. $32\frac{56}{59}$in. (39) £594. (40) $52\frac{1}{2}$ acres worth £210 16s. $2\frac{1}{4}$d. (41) £85 1s. $11\frac{1}{2}$d., and 4416·16 dollars. (42) 145 lb. 2 oz. 8 dwt. gold, 78 lbs. 6 oz. 6 dwt. silver, 67 lbs. 1 oz. 10 dwt. copper. (43) 128·8659 years. (44) 116·132163 sq. ft.; 4185·3832775 sq. in.; 5·93951 sq. yds. (45) 1st, 9515; 2nd, 11418; 3rd, $11893\frac{3}{4}$. (46) At 6 mils he would lose £1·007; at 7 mils he would gain £3·021. (47) Present receipts £66 1s. 8d.; receipts at the decimal rates £79 6s. (48) 3·2616083 ft.; $1\cdot087202\dot{7}$ yds.; ·00494183 fur. (49) 248·31441519 ft. (50) 69·3185 in.; 34·65925 in.; 8·6648125 in. (51) 53·93°. (52) 1193·174 oz. (53) £60. (54) Sum = ·01110201; Differ. = ·00910009; Prod. = 0000101111. (55) £16·222. (56) $1\frac{1}{11}$ hours. (57) $\cdot8\dot{3}$. (58) £7297 1s. $9\frac{3}{4}$d.; £16662 6s. $9\frac{3}{4}$d. (59) $2\frac{2}{5}$ days. (60) $492\frac{1}{2}$ grs. (61) $\frac{4}{7}$, $2\frac{14}{25}$. (62) £3·529; £14·166; £20·966. (63) $1\cdot817\dot{2}$ miles. (64) 1·47 cwt. (65) $\frac{182}{375}$. (66) $2096\frac{37}{70}$. (67) $261\frac{231}{425}$.

XCIX.—(1) 20. (2) 7. (3) 204. (4) 1653. (5) $3597\frac{2}{15}$ (6) $24\frac{1}{4}$. (7) 16640 8) 450. (9) 250. (10) $4147\frac{19}{31}$.

CI.—(1) 2 : 11; 25 : 48; 11 : 126; 7 : 15. (2) 117 : 296; 27 : 1624; 5 : 12; 1 : 9. (3) 49 : 31; 107 : 117; 54 : 101; 11 : 125.

CIII.—(1) $185\frac{695}{878}$ yds. (2) £8131 15s. $3\frac{9}{17}$d. (3) 1127 19s. $9\frac{11}{217}$d. (4) 9 lbs. $12\frac{4}{5}$ oz. (5) $123\frac{37}{81}$. (6) 400 and 1400. (7) £5 0s. 10d.; £4 10s. 9d.; £4 0s. 8d.; £3 10s. 7d. (8)

4·95 miles; 121·82 yds. (9) £2 1s. $9\frac{3}{4}$d. $\frac{1}{25}$. (10) $5\frac{3}{4}$d. $\frac{301}{32505}$. (11) 1s. $7\frac{1}{4}$d. $\frac{23}{25}$. (12) 70, and $2\frac{3}{16}$. (13) £88 0s. $4\frac{88}{95}$d. (14) $11\frac{1}{14}$d. (15) 7s. $11\frac{288}{333}$d. (16) $134\frac{1}{6}$ miles; $11\frac{2}{3}$ hours. (17) 847 : 900. (18) $555\frac{407}{10080}$. (19) £52 18s. (20) 97 lbs. $11\frac{1509}{1617}$. (21) 2s. $1\frac{61}{105}$d. (22) $101\frac{40}{81}$. (23) 1 hour 48 min. $28\frac{28}{44}$ sec. (24) 9 acres 20 $\frac{135}{484}$ poles. (25) 6s. $1\frac{1}{2}$d. $\frac{18}{37}$; £18 0s. $5\frac{1}{4}$d. $\frac{8}{37}$. (26) £834 15s. $1\frac{11}{80}$d. (27) £9846 5s. $10\frac{3}{4}$d. $\frac{17}{61}$. (28) £168. (29) 21·3. (30) 57·448 yds. (31) 95·099 aune. (32) £86907 13s. 1d. (33) £80 16s. $7\frac{43}{271}$d. (34) 88·269 ft. (35) £9 19s. 11d. $\frac{22}{73}$. (36) £745. (37) 3 tons 15 cwt. 2 qrs. 11·09 lbs. (38) $10\frac{8}{229}$d. (39) 64.

CV.—(1) $3291\frac{3}{7}$ yds. (2) $17\frac{7}{9}$ hours. (3) 180 men. (4) 21 horses. (5) $11\frac{27}{35}$ months. (6) $16\frac{26}{119}$ days. (7) 270. (8) 20. (9) 200 days. (10) 90. (11) £613161·29. (12) £132. (13) £500 18s. $9\frac{1}{2}$d. (14) 80·12 acres.

CVI.—(1) £17 7s. $7\frac{1}{5}$d. (2) £276 12s. $3\frac{3}{5}$d. (3) £47 8s. $8\frac{44}{55}$d. (4) £87 9s. $3\frac{3}{5}$d. (5) £394 16s. $0\frac{9}{10}$d. (6) £144 6s. $4\frac{1}{2}$d. (7) £175 5s. 9d. (8) £1880 6s. $9\frac{3}{5}$d. (9) £1003 5s. 6d. (10) £78 2s. $3\frac{1}{2}$d. (11) £44 4s. $3\frac{1}{2}$d. (12) £85 12s. $5\frac{3}{4}$d. (13) £519 18s. $2\frac{1}{2}$d. (14) £646 10s. $6\frac{1}{2}$d. (15) £296 14s. $3\frac{3}{4}$d. (16) £51 18s. $1\frac{1}{4}$d. (17) £3699 14s. (18) 2148 11s. 3d. (19) £3526 6s. $11\frac{3}{4}$d. (20) 25 years 225 days. (21) 16 years 168 days. (22) 3 years 126 days. (23) 7 years 218 days. (24) £4. (25) £3 10s. $10\frac{1}{2}$d. (26) £4 5s. $0\frac{90}{127}$d. (27) £2 13s. $1\frac{1}{4}$d. (28) £860. (29) £27400. (30) £2228 11s. $5\frac{1}{7}$d. (31) £1744 10s. $9\frac{9}{119}$d. (32) 8 years. (33) £6666 13s. 4d. (34) £4. (35) $18\frac{1}{2}$ years. (36) £2826 8s. $8\frac{3}{4}$d. (37) $28\frac{4}{7}$ years. (38) £2406 6s. $2\frac{1}{2}$d. (39) £35 7s. $8\frac{1}{2}$d.

CVII.—(1) £62 8s. $7\frac{17}{25}$d.; or £62·432. (2) £101 12s. $9\frac{1}{2}$d.; or £101·640. (3) £149 11s. $5\frac{1}{4}$d.; or £149·572. (4) £90·84. (5) £154·5. (6) £713·380. (7) £129·254. (8) £205·892. (9) £404·943. (10) £349·499. (11) £216·784. (12) £191·129. (13) £170·714. (14) £278·596.

CVIII.—(1) £679 4s. $10\frac{48}{55}$d. (2) £1020. (3) £844 8s. $10\frac{2}{3}$d. (4) 5s. $8\frac{32}{41}$d. (5) £55 15s. $3\frac{1}{3}$d. (6) £1 16s. $10\frac{286}{407}$d. (7) £558 $11\frac{29}{41}$d. (8) £3802 5s. $7\frac{589}{988}$d. (9) $11\frac{1}{2}$d.; 1s. $6\frac{1}{4}$d.; 3s. $0\frac{1}{4}$d. (10) 3s. 9d.; £1 0s. $9\frac{3}{4}$d.; 2d.; or £·0079. (11) £·0028; $\frac{788544}{2678005}$ of a penny; and 1s. $5\frac{1}{4}$d. (12) £1 10s. $4\frac{1}{4}$d. (13) £7144 3s. $8\frac{28}{43}$d. (14) £5066 19s. $3\frac{3}{7}$d. (15) £31 2s. $1\frac{1}{2}$d. (16) £844 7s. $1\frac{485}{923}$d. (17) 25 per cent. (18) £926 11s. $10\frac{3}{4}$d.

CIX.—(1) £7140 16s. $2\frac{2}{5}$d. (2) £124 6s. 4d. (3) 87 $\frac{859}{1075}$. (4) £62 1s. $6\frac{6}{53}$d. (5) $91\frac{5}{8}$. (6) £5 12s. $8\frac{1}{4}$d. (7) £19 0s. $7\frac{65}{191}$d. (8) £7352 15s. $5\frac{11}{20}$d. (9) £1026 11s. 3d. (10) £27 2s. 6d. (11) $104\frac{1}{3}$. (12) £118 4s. $6\frac{78}{203}$d. (13) £841 12s. 10 $\frac{38}{73}$d. (14) 97·243. (15) £3·5278. (16) $87\frac{1}{2}$. (17) £·3605 per cent. (18) £179 2s. 5d.

CX.—(1) 110 in geography, 90 in grammar, 30 cannot read, and 10 in algebra. (2) £2072 14s. $6\frac{6}{11}$d. (3) In the first, 617·5; and in the second, 314·5. (4) 2473·507 ft. nitrogen; 763·4968 ft. oxygen; 36·9962 ft. carbon. (5) 14 passengers, 55 railway servants, and 29 trespassers. (6) ·0000358. (7) 14·52; 12·65; 11·93; and 13·19. (8) 12·2262. (9) 42592; 81808; and 14516. (10) In the public, 79·23; and in the private, 91·27. (11) 11·75. (12) £173 0s. $4\frac{1}{2}$d. (13) £2250. (14) £2360·51. (15) Cost 3s. $1\frac{1}{2}$d., gain 8 per cent. (16) $8\frac{148}{369}$. (17) Total gain = £1 19s. 4d., and gain per cent. = $21\frac{83}{277}$ (18) £450. (19) 5s. 3d. (20) £668 9s. $2\frac{91}{100}$d. (21) £4098 7s. $2\frac{34}{81}$d.

MISCELLANEOUS EXERCISES IN PROPORTION.

(1) 15. (2) £810 16s. $0\frac{1}{2}$d. (3) £2 3s. $6\frac{1}{4}$d. (4) 7200 soldiers. (5) £2 0s. 9d. (6) 2s. $4\frac{401}{1030}$d. (7) £911 less £905·974025 = £5·025974. (8) £5 12s. $1\frac{1}{11}$d. (9) 232 0s. $7\frac{19}{23}$d. (10) $10\frac{1}{4}$d. (11) 116 13s. 4d. (12) £6 13s. 4d. (13) £402 10s. (14) £16 9s. $5\frac{1}{2}$d. (15) £125 7s. $10\frac{7}{25}$d. (16) 4 cwt. 2 qrs. $17\frac{97}{108}$ lbs. (17) £70 19s. $6\frac{222}{317}$d. (18) 12 hours. (19) 6s. $1\frac{1}{23}$d. (20) £3 5s. $11\frac{19}{31}$d. (21) £463 6s. $4\frac{4}{5}$d.; £453 19s. $2\frac{2}{5}$d.; £421 4s. (22) £1250. (23) £3 2s. $4\frac{76}{113}$d. (24) £960 12s. 8d. (25) 12s. 6d. in the pound; £172 17s. $2\frac{1}{4}$d. (26) £589 6s. $2\frac{1}{4}$d. $\frac{4}{5}$. (27) £821 18s. $6\frac{1}{2}$d. (28) £1 15s.; $8\frac{1}{3}$ per cent. (29) £67 2s. $10\frac{542}{637}$d. (30) £3 2s. $5\frac{1909}{2951}$d. (31) A £17 14s. $11\frac{11}{71}$d. B £17 5s. $0\frac{3}{4}$d. $\frac{27}{71}$. (32) 18 men. (33) He will gain £75 by the latter. (34) 20·3326 minutes. (35) $11\frac{9}{11}$ years. (36) $22\frac{2}{9}$ per cent. (37) 54; 81; 108; 135; 162. (38) $30\frac{10}{13}$ years. (39) 9s. $6\frac{1}{4}$d. $\frac{1}{7}$. (40) £124 10s. $2\frac{8}{11}$d. (41) 160. (42) As 40 : 41; £123 10s. 3d. (43) 146·35 and 134·903. (44) £1388·88. (45) 4 months 1 week 4 $\frac{1}{13}$days. (46) £3 12s. $3\frac{1}{2}$d. (47) £397 17s. 6d. (48) £936 15s. $4\frac{1}{4}$d. + $\frac{181}{427}$ (49) £11·11 per cent. (50) 11 cwt. 3 qrs. $11\frac{106}{133}$ lbs. (51) 10 weeks 5 days 20 hours. (52) 600. (53) $740\frac{4}{7}$ days. (54) £6 9s. $11\frac{1}{4}$d. (55) $8\frac{19}{22}$ seconds. (56) 2·7554 per cent. (57) $7\frac{1}{2}$ ft. (58) £4 15s. $9\frac{3}{4}$d.

(59) £301 10s. (60) 7 hours 10 min. 17$\frac{1}{7}$ sec. (61) 424107 tons 2 cwt. 3 qrs. 12 lbs. (62) 54 to 275. (63) £8 5s. 1$\frac{1}{2}$d. (64) 8·395d.

CXXIII.—(1) 4207; 835·754; 78; 70·0428. (2) 29; 35·874; 7538; 845. (3) 3096; 213; 104; 27·3. (4) 58·06; 270·2; ·016; $\frac{15}{16}$. (5) 2·401; 317·8; 217. (6) 610·2; 52·615; 6301·244. (7) 120·79; 3·14; ·0769. (8) 3·825; 4·093; ·082. (9) ·63245; ·9826073; 1$\frac{1}{2}$; 3·9749. (10) 2·5298; ·8; ·25298; ·08; 25·298. (11) 3·5496; 8·01249; 4·27434; 6·4. (12) ·95452; 48·6698; ·3902436; ·790936. (13) 3$\frac{4}{15}$; 14$\frac{27}{40}$. (14) 8$\frac{13}{20}$; $\frac{23}{30}$. (15) 23$\frac{24}{35}$; 20$\frac{8}{25}$. (16) 7$\frac{14}{31}$; 4$\frac{7}{11}$. (17) 30·0202. (18) 714. (19) 2·8284, and 4·1683.

CXXVIII.—(1) 32; 10·5; 628. (2) 319; 4283·77; 1·72. (3) 1·09; 1·44; 16·1; 51·4. (4) 2009; 5493·612. (5) 1·077; 1·442; 130·011; ·7631. (6) 1375; 6281. (7) 7·092; 602·8. (8) 5·172; 37·24. (9) $\frac{14}{15}$; 3$\frac{8}{25}$. (10) $\frac{23}{30}$; $\frac{41}{45}$. (11) 2$\frac{13}{15}$; 6$\frac{15}{32}$. (12) a. 90·03; 80·7; 18·45. b. ·843; ·879; ·4028; 3·001. c. 2·59; 2·636; 7·988; 3·976. (13) 1472.

CXXIX.—(1) 45 ft. 3′ 10″ 10‴ 8⁗, and 6 ft. 2′ 1″ 5‴ 5⁗. (2) 355 ft. 1′ 8″ 3‴ 9⁗, and 311 ft. 5′ 2″ 8‴. (3) 1620 ft. 5′ 7″ 11‴) and 383 ft. 0′ 6″ 8‴ 3⁗. (4) 104 ft. 1′ 5″ 3‴ 5⁗ 3′′′′′, and 32·30593 ft. (5) 35·82825 yds., and 14652·97242 ft. (6) 4100$\frac{8}{15}$, and 6488$\frac{17}{32}$. (7) 87·885 ft. (8) 10·147 ft. (9) 10·26596. (10) 332 ft. 3′ 9″ 6‴ 4⁗ 6′′′′′. (11) 22 ft. 11′ 7″ 5‴ 3⁗. (12) 8 ft. 9′ 9″. (13) 16 ft. 7′. (14) 14 ft. 4′. (15) 45$\frac{1}{8}$. (16) 4 yds. 2 ft. 9$\frac{3}{7}$′. (17) 34 ft. 4′ 6″, and cost = £1 1s. 1$\frac{3}{4}$d. (18) £51 4s. (19) 785 ft. 9′ 4″. (20) 6′ 4″. (21) £5 17s. 11$\frac{5}{48}$. (22) £11 1s. 2$\frac{7}{16}$d. (23) 199 yds. 1$\frac{59}{108}$ ft. (24) 1650 : 1633. (25) 4483750 : 5602689. (26) 716831 : 549654. (27) 702 square ft. (28) £2287 14s. 2d. (29) 11$\frac{591}{1022}$ ft. (30) £2 13s. 7$\frac{49}{72}$.

CXXX.—(1) 16·5529, and 18·357. (2) 25·819. (3) 24·392, and 138·971. (4) 1 ft. 9·768′. (5) 10 ft. (6) 19·798, and 29·132, and 46·669. (7) 17·677; 40·729; 60·386. (8) 24·624675. (9) 70·5 yds. (10) 506·835. (11) The first is double the second. (12) 62500 : 150709, and 88209 : 150709. (13) 45·131 leagues. (14) 143·443. (15) 71·299, and 4·219. (16) 15·909: 20·859.

CXXXI.—(1) 53·40703; 168·389224; 775·97273; 34·243331. (2) 49·019; 66·399; 1291·702. (3) 7912·032 miles. (4)

34·377. (5) 3 ft. 3·3723 in. (6) 36° 25′ 45·9″. (7) 280·112 ft. (8) 122·522. (9) 79·848. (10) 394 ft. (11) $14\frac{56}{171}$ ft. (12) 32° 13′ 13″. (13) 1210·125 miles.

CXXXII.—(1) 153·93791, and 201·06176, and 254·46879. (2) 4417·8609375; 5089·572149375; 40·60198769. (3) 6·2 ft. (4) 124·12 yds. (5) 289 : 324 : 361. (6) 5326·237224. (7) 889·8582 ft. (8) 177·2321. (9) 30·7035 ft. (10) 3899·3 acres. (11) 6·35 acres. (12) £16 14s. 11¼d., and £60 15s. 5¼d. (13) 17s. 8d.

CXXXIII.—(1) 6708 ft. 3′ 3″. (2) 1391 ft. 1′ 10″ 6‴, and cost £23 3s. 8½d. nearly. (3) 57375 oz. (4) £531 14s. 6⅔d. (5) 118800. (6) 3 tons 16 cwt. 19¼ lbs. (7) 3 tons 5 cwt. 2 qrs. 4 lbs. 10 oz. (8) £6852 18s. 9d. (9) £4 0s. 6¾d., weighs 8 cwt. 1 qr. 20⅜lbs. (10) 1411141. (11) £68 18s. 8⅓d. (12) 5·357 ft. (13) 2·8409 miles per hour.

CXXXIV.—(1) 5038509·7997, and 8379·174. (2) 3346·562, and 260703610340 miles. (3) 1795899·426 (4) 75729·740625. (5) 1875·789. (6) 4206·6228. (7) 1·5217125. (8) 2 ft. 8′. (9) 14·85 ft. (10) 9199·0760263045875. (11) 45 ft. 3′ (12) 263·8869.

(1) 182250. (2) 4·5 ft. (3) £25 7s. 2d. (4) 3·14159. (5) 61·842 ft.

CXXXV.—(1) 71. (2) 234. (3) 40. (4) 49. (5) 401. (6) 182. (7) 39. (8) 80. (9) 97. (10) 47. (11) 8. (12) 64. (13) 15. (14) 26½. (15) 26½.

CXXXVI.—(1) 7; 9; 11; 13; 15; 17; 19; 21; 23; 25; 27; 29; 31; 33; 35. (2) 15; 22; 29. (3) 8; 11; 14; 17; 20; 23; 26; 29; 32; 35; 38; 41; 44; 47; 50; 53. (4) 72; 138; 204; 270; 336; 402; 468; 534.

CXXXVII.—a. (1) 54, 57, 70, 119. (2) 51, 80, 67, 73. (3) 54, 60·5, 9·83. (4) 1, $\frac{169}{180}$, $\frac{354}{775}$. (5) $5\frac{21}{80}$, $31\frac{21}{216}$, 27. b. (1) 19, 18, 27⅓. (2) 5⅓, 9½, 22¾. (3) 14¾, 9, 29½. (4) 57⅝, 8·9, 4·683.

CXXXVIII.—(1) 507, 39700. (2) 558, 625. (3) 851, 19307. (4) 94⅓, 175. (5) 288¾, 443½. (6) 800, 2420. (7) $13\frac{5}{7}$, 0. (8) 115, 45. (9) 1092. (10) £8 16s. (11) 21 days. (12) 12502497. (13) £3 7s. 9d. (14) 531·3 ft.

CXXXIX.—(1) 576, and 8649. (2) 1875. (3) 112338. (4) 171125. (5) 2187·5. (6) 3468. (7) 5776 feet.

CXLI.—(1) 49152. (2) 2048. (3) 4374. (4) 1792. (5) $\frac{2}{512}$. (6) 81920. (7) 3000000. (8) $\frac{2}{243}$. (9) 354294. (10) $\frac{3}{128}$. (11) 1220·7. (12) 524288. (13) $\frac{1}{256}$. (14) ·125. (15) $1\frac{7}{9}$. (16) $\frac{3}{2048}$. (17) $\frac{128}{1215}$.

CXLII.—(1) 54, 203, 18. (2) 60, 90, 143. (3) 6, 40·124, 984·69.

CXLIII.—(1) 5461, $39\frac{9}{16}$. (2) $15\frac{6487}{65536}$, $4\frac{54953}{118098}$. (3) 109225, $166\frac{50718}{78125}$. (4) 1, ½. (5) 72 miles. (6) 3752999689 4754 bush. 1 gal. 1 pint. (7) 109225.

CXLIV.—(7) $\bar{1}$·7552992. (8) 1·8976335. The remainder of the answers in this exercise will be found in books of tables.

CXLV.—(1) 45·41389; ·000556561. (2) 917·627; 867055. (3) ·5700212; ·06920857. (4) 1·47487653; 1·2956932. (5) 1·902222; 32·0524648. (6) 8192·0019; ·42382077. (7) ·0106695503; ·124975. (8) ·1360485; 75·044. (9) ·0018271; 9·658176. (10) 112030; 5·79. (11) 6·25; 111·567. (12) 1·1115; 43·93. (13) 14·437. (14) 65; 937·5. (15) 14·12. (16) 1·70188; 41·716. (17) ·295784; 144·5972.

CXLVI.—(1) £9057 8s. 5d. (2) £1416 6s. 4d. (3) £27094 10s. 10½d. (4) £2072 10s. 4½d. (5) £15192 18s. 10½d. (6) £21673 6s. 8d. (7) £313 13s. 4½d.

CXLVII.—(1) £3342 6s. 11¾d. (2) £482 11s. 1d. (3) £1173 17s. 2¼d. (4) £13299 16s. 4¾d. (5) £2032 0s. 0¼d. (6) £9 5s. 11d. (7) £2482 12s. 3d. (8) £7373 7s. 11d. (9) £4127 5s. 1d. (10) £13321 1s. 10½d. (11) £8 17s. 0¾d. (12) £1298 0s. 6½d. (13) £407 2s. (14) £3978 12s. 3½d.

CXLVIII.—(1) £2118 11s. 6d. (2) £2715 8s. 5d. (3) £359 4s. 0¾d. (4) £2628 15s. 2¼d. (5) £1790 13s. 9¾d. (6) £1208 7s. 9½d. (7) £7465 1s. 5¾d. (8) £1604 13s. 4¾d. (9) £7682 1s. 2d.

CXLIX.—(1) ·13572; 184·1847; ·911899; 1084244250000 0. (2) 15·9 years. (3) 2·2146; ·30103; 1·537243. (4) 47·141 years. (5) 1, ½, and ¼. (6) ·1757. (7) 20 years. (8) 30 years 9 months 3¾ weeks. (9) 4 years. (10) £8209 14s. 1½d. (11) ·19195; 127·097; ·028719; ·71588; 3462·1528; 325·35187; 3·1798. (12) 1·766. (13) ·454. (14) 1575649454545454. (15) 30686·8208296 days. (16) 77·898 years. (17) £6 12s. 5¼d. (18) 1½ years. (19) 36·497 yrs. (20) £255 11s. 5d. (21)

·7 per cent. (22) £5 10s. $6\frac{1}{2}$d. (23) 2630583·6. (24) 1142·26[illegible]. (25) £19 0s. 8d. (26) £1322 4s. $4\frac{3}{4}$d. (27) 1·7949. (28) £1755 6s. $0\frac{1}{2}$d. (29) £7746 17s. $6\frac{1}{4}$d. (30) £13 0s. $3\frac{1}{4}$d. (31) £173 13s. 7d. (32) £17201 2s. $0\frac{1}{4}$d. (33) £2972 17s. $4\frac{3}{4}$d. (34) 128, 1024, 8192.

CL.—(1) 3133424. (2) 100111[illegible]00010000; 41104; 7571. (3) 4861. (4) 13662. (5) 3392. (6) 5. (7) 31a 0. (8) 60242; 36836; 1764a.

MISCELLANEOUS QUESTIONS.

(1) $5\frac{3}{8}$. (2) £416 19s. $4\frac{1}{2}$d. (3) $93\frac{9}{10}$. (4) 883345·172192 miles. (5) £1499·78. (6) (*a*) £18 0s. $10\frac{3}{8}$d.; (*b*) £1 13s. 1d. (7) Julian year 11 min. 10·3 secs. too long, Gregorian 22·3 secs. too long. (8) 13 days 13 hours 50 mins. 25 secs. (9) £13. (10) £116 13s. 4d. per cent. (11) 159 square feet 7′ 3″. (12) 82554. (14) $3\frac{65}{72}$. (15) $\frac{9}{1580}$; $\frac{81}{7900}$ of an inch. (16) £1338·837. (17) ·120508; ·13. (18) ·0272. (19) 508·3. (20) $\frac{a+b}{2}$ arithmetical, $\sqrt{ab}$ geometrical; 62, $\frac{211}{243}$. (21) 16000. (22) 43 yards 3 quarters. (23) Loses $6\frac{2}{3}$d. (24) £2.934. (25) A's share £300, B's £900. (26) £500 and £5000. (27) ·22433. (28) £3 2s. 3d. (29) $34\frac{2}{7}$. (30) $\cdot 333\dot{1}\dot{3}$. (31) (*a*) 33·022436. (*b*) 125·37513. (*c*) 67·58. (*d*) 50000. (*e*) 46·04. (32) 17s. $5\frac{3}{8}$d. (33) 288; 196605. (34) 104 days. (35) 72 miles. (36) 18 men. (37) 18s. 3d. (38) 1936. (39) 13·2023 dozen. (40) £231 8s. (41) 3 square feet 19 square inches. (42) £5 2s. $4\frac{1}{2}$d. (43) £33 15s. (44) 84. (45) Man's share £1 10s., woman's £1 2s. 6d., and child's 12s. (46) $27\frac{1847}{2365}$. (47) 143969888816·065 lbs. (48) (1.) $67178\frac{4}{7}$. (2.) $10\frac{870}{927}$. (3.) 7 hours 51 minutes 40 seconds. (49) Annual profit £9268 2s. 7d.; of which Government share £3450 14s. 7d., and colonists £5817 7s. 11d. (50) $89\frac{5}{27}$ miles. (51) £13 14s. 2d. (52) $1\frac{11}{25}$ months. (53) 18. (54) $\cdot\dot{3}52941176470588\dot{2}$. (55) £5 12s. 6d. per annum. (56) 26. (57) 37 cwt., 3 qrs. (58) 3·35347 inches. (59) 107 men and 1 boy. (61) 9·0636. (62) $38\frac{4}{7}$ per cent. (63) 3, 3, 2, 2, 5, 7, 43. (65) £563·243. (66) 54. (67) 3s. $0\frac{3}{8}$d. per yard. (68) ·8499. (69) £2 nearly. (70) £53 13s. 4d. (71) 687 tons, 1 cwt., 8 lbs. (72) 2·64575 and 8·0259. (73) $2\frac{1}{2}$ per cent. (74) $\frac{5}{21}$. (75) £17 2s. 6d.

(76) $21\frac{7}{33}$. (77) $4\frac{1}{2}$ per cent. (78) $1714\frac{2}{7}$ and 4s. 7d. (79) 48. (80) ·04 and ·094. (81) £42·813. (82) 5376. (83) $6\cdot\dot{7}14285\dot{7}$. (84) $\cdot\dot{7}1428\dot{5}$; $\frac{71}{99}$; $\frac{40287}{49950}$. (85) Log. 80 = 1·90309; log. 81 = 1·9084852; log. 360 = 2·5563026. (86) 3, 234, 240. (87) £10236 13s. $8\frac{1}{2}$d. (88) $115\frac{1}{3}$. (89) 15248787840 grains. (90) £1450 4s. (91) A 3s. 6d., B 4s. 8d., C 5s. 10d., and D 7s. (92) £26149·588. (93) 525 miles by water, 175 by railway, and 105 on foot. (94) £102·25. (95) £644 18s. 9d. (96) 18·7494 feet. (97) 914·6644 (98) £609 1s. 7d.

ANSWERS TO OXFORD AND CAMBRIDGE EXAMINATION PAPERS, p. 350.

OXFORD.

SENIOR STUDENTS I.—(1) £95. (2) £37 13s. $2\frac{1}{2}$d. (3) £977 14s. $0\frac{3}{4}$d.; £12 2s. $1\frac{1}{2}$d. (4) $\frac{8}{107}$; $4\frac{1}{6}$. (5) £1. (6) ·0004; 4000; 40. (7) $\frac{3}{8}$; $\frac{8}{80}$; $\frac{109}{9990}$. (8) $\frac{1}{880}$; ·001136. (9) 7014; ·04. (10) Simple £14·28; Comp. £14·858. (11) 3738. (12) 864; £233 17s. 9d. (13) 21 days.

SENIOR STUDENTS II.—(1) 408 lbs.; 2d. (2) £49 19s. $11\frac{3}{4}$d.; 1 ton 1 cwt. 26 lbs. 15 oz. (3) £901 5s. and £10 17s. $4\frac{1}{2}$d. (4) $\frac{29}{31}$; 2. (5) 10s. (6) ·12; 24; 9. (7) $\frac{1}{80}$; ·003472. (8) 1·124864; 29496. (9) Simple £1 4s. 11·52d.; Comp. £1 5s. 11·66d. (10) Equal. (11) 96 yards. (12) 18 men.

SENIOR STUDENTS III.—(1) 90000. (2) £9 0s. $11\frac{3}{4}$d.; 81 miles 6 fur. 39 poles. (3) £650 16s. 8d.; £10 3s. $4\frac{1}{4}$d. (4) 1. (5) 19s. $9\frac{17}{20}$d. (6) 30; ·03; 3. (7) ·0505. (8) 29897; 1·21550625. (9) Simple £4·3; Comp. £4·633. (10) 25. (11) £11 6s. $0\frac{13}{28}$d. (12) 14.

JUNIOR STUDENTS I.—(1) 800; ·08; 8. (2) $\frac{1}{4}$; $\frac{1}{40}$; $\frac{7}{66}$. (3) 1. (4) ·0234375. (5) 709; ·03. (6) £1869. (7) £3 5s. 10d. (8) 8. (9) $2\frac{274}{363}$; $\frac{250}{363}$ and $2\frac{24}{363}$ acres. (10) £11 0s. 6d. and £7 12s. 3d. (11) 3·20287 cub. ft. and 19s. $2\frac{1}{2}$d. (12) 381·7044 cub. inches.

JUNIOR STUDENTS II.—(1) 708778. (2) 91132497 and 506909. (3) £1111 11s. $11\frac{3}{4}$d., and 1 ton 16 cwt. 3 qrs. 23 lbs. (4) £158 6s. $10\frac{1}{4}$d. (6) 418 tons 2 qrs. 12 lbs. 15 oz. (7) £2 2s. $2\frac{1}{4}$d. and £33 0s. $11\frac{1}{4}$d. (8) 720. (9) £2 5s. 9d. (10) £3 6s. per qr.

JUNIOR STUDENTS III.—(1) 130; 1·3; 130. (2) £1. (3) $\frac{11}{11}$; 4. (4) 1·157[illegible]5; 247009. (5) Simple £1 11s. 6d.; Comp. £1 13s. [illegible] (6) £24. (7) 12 days. (8) 6 feet. (9) £4 4s (10) 168 cub. in.; 765. (11) £2 5s. 3d. (12) ·012923654 cub. in.

CAMBRIDGE.

SENIOR STUDENTS I.—(2) £3 4s. 7¼d. (3) $16\frac{9}{20}$; 1; 1. (4) £990 16s. 10½d. (5) 98. (6) £1; ·1390625. (7) £9. (8) ½. (10) 30 min. 21 seconds past one.

SENIOR STUDENTS II.—(1) 79. (2) 761 + 16. (3) $187518772\frac{1}{2}$; 16 tons 17 cwt. 2 qrs. (4) $2\frac{4339}{10200}$; $13\frac{3}{110}$; $1\frac{13}{80}$. (5) £4011 6s. 6¾d.; (6) ·82008; ·002; 200; 21; 20000. (7) ·404761; 519·48 yards. (8) 2 tons 1 cwt. 2 qrs. 17 lbs. (9) 2 men. (10) $15\frac{5}{8}$. (11) £14520. (12) 3·60551; 3 square ya[illegible] 13·5 square inches.

JUNIOR STUDENTS I.—(1) 796·12364. (2) Lost £1 0s. 1d. (3) £1100 11s. 9¼d. (4) $1\frac{9}{20}$; $10\frac{1}{10}$. (5) £1439 18s. (6) ·4206; 501000; ·0501; ·05. (7) 66 days. (8) 10s. (9) £21 11s. 11¼d. (10) 24574 cubic feet; 1584 cubic inches. (11) £5. (12) Loss £180.

JUNIOR STUDENTS II.—(1) 1090792. (2) 2666 poles 2 feet 3 inches. (3) £2 19s. 10¼d. (4) 5 times, and £13 11s. 0¼d. remainder. (5) $\frac{43}{102}$; 25. (6) ·2; 200; ·00002. (7) £38071 1s. 3d. (8) $12\frac{1}{8}$ days. (9) 4400. (10) £450 and £484·59375. (11) $1\frac{1}{9}$; $1\frac{4}{5}$; $\frac{4}{5}$. (12) £130. (13) Gained £32 16s. 8d.

J. AND W. RIDER, Printers, 14, Bartholomew Close, London.